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Taken from a Brass Coin in the Repository of the late Queen Christian of Swede

Euclid *the Mathematician, was of* Alexandria, *where he taught in the Reign of* Ptolemy Lagus *in the* CXX Olympiad, *and Year of* Rome 454. *He Wrote many things relating to Musick and Geometry: But his* XV *Books of Elements (of which he is generaly thought to be only the Collector) are most applauded: the two last are attributed to* Hypsicles *of* Alexandria, *and not to him.* Cardan Vossius.

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Geo. Wm Briggs. 1824

THE Elements of *EUCLID*; WITH Select THEOREMS OUT OF *ARCHIMEDES*.

By the Learned ANDREW TACQUET.

To which are added,

Practical COROLLARIES, shewing the Uses of many of the Propositions.

The Whole abridg'd, and in this THIRD EDITION Publish'd in *English*,

By WILLIAM WHISTON, M. A. Mr. *Lucas*'s Professor of the *Mathematicks* in the University of *Cambridge*.

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AN Historical Account OF THE RISE and PROGRESS OF THE MATHEMATICKS.

IT seem'd meet to me when I was about to set forth the Elements of the Mathematicks, to premise a few Things concerning the Rise and Excellency of this Science, that its Candidates may understand what a Kind of Science it is to which they are about to dedicate themselves; and that it may be made manifest against those who slight those Things whereof they are ignorant of how great Value and Dignity this Knowledge is, which the wisest Men of all Ages have, with incredible Study, labour'd to attain unto, and become possess'd of. Moreover, I must own that *Peter Ramus*'s Labours have been of great Service to me in the compiling

of this Account, who in the whole first Book of his Institution, which is not a little one, hath out of *Proclus*, *Laertius*, *Gellius*, *Polybius*, *Tzetzes*, and others, compos'd a Mathematical History both accurately and copiously.

The Mathematical Sciences were the first of all other amongst Men, if we may believe *Josephus*. He, *Book* 1. *Chap.* 3. writeth, that the Posterity of *Seth* observ'd the Order of the Heavens, and the Courses of the Stars. And lest these Inventions should slip out of the Knowledge of Men, *Adam* having predicted a twofold Destruction of the Earth, one by a Deluge, the other by Fire, they rais'd two Columns, one of Brick, of Stone the other; and inscribed their Inventions upon them, that if the Brick one should happen to be destroy'd by the Deluge, that of Stone which would remain might afford Men an Opportunity of being instructed, and present to their View the Things which it had inscribed on it. They say also that that stone Pillar, which even in our Days is seen in *Syria*, was dedicated by them. This *Josephus* says: whom I leave to vouch for the Story.

That the *Assyrians* and *Chaldeans* were the first of Mortals, after the Flood, who applied themselves to the Mathematicks, is delivered by the same *Josephus*; as also by *Pliny*, *Diodorus*, and *Cicero*. But the Mathematick Arts which first sprang amongst the *Chaldeans*, amongst whom they flourished, were afterwards transferr'd out of *Chaldea* and *Assyria* unto the *Egyptians*, by *Abraham*. For, when,

when, at the Command of God, he went forth from his native Soil into *Paleſtine*, and from thence into *Egypt*, and perceiv'd the *Egyptians* to be taken with the Study of good Arts, and to be of a very notable Wit and Capacity for Learning, (as *Joſephus* teſtifies, *Book* I. *Chap.* 9.) ; he communicated to them Arithmetick and Aſtronomy; and conſequently Geometry, which muſt of Neceſſity go before Aſtronomy. In which Studies afterwards the *Egyptians* ſo flouriſh'd, that *Ariſtotle*, 1 *Metaph. Chap.* 1. doth affirm, That the Mathematick Arts were firſt found out in *Egypt*, by their Prieſts ; who by their Employment were at leiſure for theſe Things.

Then theſe Arts croſſing the Sea out of *Egypt*, came to the Philoſophers of *Greece*: For *Thales* the *Mileſian*, who flouriſh'd 584 Years before Chriſt, was the firſt of the *Greeks*, who coming into *Egypt*, transferr'd Geometry from thence into *Greece*. He it was indeed, who, beſides other Things, found out the 5th, 15th, and 26th, Propoſitions of the firſt Book. To the ſame are alſo owing the 2d, 3d, 4th, 5th, of the Fourth Book. The ſame Perſon began to obſerve the Equinoxes and Solſtices, as *Laertius* teſtifies ; and he was the firſt who foretold an Eclipſe of the Sun, as *Hippias* and *Ariſtotle* do write ; and *Tzetzes* ſaith, That he alſo foreſhew'd an Eclipſe of the Moon to King *Cyrus*. For which Things ſake he is to be look'd on as the firſt Founder and Author of the Mathematical Sciences in *Greece*.

After him was *Pythagoras* of *Samos*: Which moſt ancient Philoſopher, exceedingly im-

prov'd and adorn'd the Mathematick Sciences. And he so gave himself to Arithmetick in particular, that almost his whole Method of Philosophizing was taken from Numbers. And he first of all, as *Laertius* relates, abstracted Geometry from Matter; in which Elevation of the Mind, he found out the 32d, 44th, 47th, and 48th Propositions of the First Book. But he is especially celebrated for the Invention of *Prop.* 32, and 47. of that Book; and he conceiv'd so great Joy upon this Invention, that, as *Apollodorus* witnesses in *Laertius*, on that Account he sacrific'd an Hecatomb. The same Person first laid open the Matter of incommensurable Magnitudes, and the Five regular Bodies. The same Person did both most diligently teach and exercise the Art of Astrology and Musick: For he did not only acutely and subtly find out many Things himself, but he also first opened a School, in which Youth might learn these honourable and noble Arts.

Pythagoras was followed by *Anaxagoras* of *Clazomenæ*, and *Oenopides* of *Chios*, of whom *Plato* makes mention in his Dialogue, *The Lovers*, where Young Men are brought in contending about *Anaxagoras* and *Oenopides* in their Descriptions of Circles. *Aristotle* reports, that a certain Treatise of Geometry was written by *Anaxagoras*; and we have it from *Laertius*, that it was shew'd by him that the Sun is greater than *Peloponnesus* (a notable Instance of the Infancy of Astronomy at that Time); and that he made some Conjectures concerning Habitations in the Moon. As for *Oenopides*, to him *Proclus* ascribes the 12,

 and

and 13. *l.* 1. These were followed by *Briso*, *Antipho*, and *Hippocrates* of *Chios*, all of them, for attempting the Quadrature of the Circle, reprehended by *Aristotle*, and at the same time celebrated. But amongst them, *Hippocrates* was by far the most Famous; that Celebrated Person, who of a Merchant growing to be a Philosopher and Geometrician, besides the Quadrature of the Circle, also first attempted the Doubling of the Cube, by Two mean Proportionals, which as being an Excellent, and indeed the only Way, all that have follow'd him to this time have embrac'd it. 'Tis also his peculiar and great Commendation, that he, as *Proclus* testifies, first wrote Elements, and digested into Order the Discoveries made by others.

Democritus was Admirable, not in Philosophy only, but also in the Mathematicks. His Physical Monuments, and, if such there were, his Mathematical Works also, are wholly lost, through the Envy (as some report) of *Aristotle*, who desired to have no other Writings read but his own. The Philosophy of *Democritus* hath been restored by *Peter Gassendus*, in a most Learned Work lately put forth. *Theodorus Cyrenæus*, although none of his Mathematical Inventions are extant; yet is great upon this Account, if there were no other, that he is reported to have been the Master of *Plato*.

Unto *Plato* therefore we are come at length, than whom no one brought greater Lustre to the Mathematical Sciences. He amplified Geometry with great and notable Additions, bestowing incredible Study upon it. And

above all, the Art Analytick, or of Resolution, was found out by him, the most certain way of Invention and Reasoning. He set off and illustrated his Books of Philosophy in a Mathematical way, and encourag'd whatsoever was admirable in Mathematical Philosophy. Upon the Door of his Academy was read this Inscription: οὐδεὶς ἀγεωμέτρητος εἰσίτω: *Let no one ignorant of Geometry enter here*; an Illustrious Instance to demonstrate, how the Mathematicks are not foreign but proper, not unuseful, or unbecoming, but honourable and profitable to sound and certain Philosophy. In a word, how great both Admirer and Master of the Mathematicks *Plato* was, that Man will of himself easily understand, who shall read his Monuments through.

Out of *Plato*'s Academy, almost innumerable Mathematicians came forth. Thirteen of *Plato*'s familiar Acquaintance are commemorated by *Proclus*, as Men by whose Studies the Mathematicks were improv'd. From hence were *Leodamus* the *Thasian*, *Archytas* the *Tarentine*, *Theætetus* the *Athenian*, by whom the Mathematicks were notably enlarged. *Leodamus* practised the Analysis received from *Plato*, and is said by *Laertius* to have found out many things by the Help of it. As for *Theætetus*, both his own Inventions, amongst which are the Elements written by him, and the Inscription of regular Bodies; and *Plato*'s Encomiums, who also inscribed a Dialogue to his Name, do make him famous.

Archytas also wrote Elements himself; and his doubling of the Cube is mentioned by *Eutocius*; whose singular Commendation it like-

likewiſe was, that he was almoſt the Firſt that brought down the Mathematicks to human Uſes; by whoſe Contrivance alſo a wooden Pigeon was made to fly, as *Gellius* reports; he being followed by *Dædalus*, and other Artificers, yielded Matter for the Fables of the Poets. Moreover, *Archytas* was both a Mathematician and General of an Army: He five Times commanded the Forces of his own Citizens, in the Wars of his Countrey, and five Times overcame their Enemies. The meer Name of *Neoclides* is only Famous, he being more illuſtrious for his Scholar *Leon* perhaps, than for his own Inventions. *Leon* certainly wrote Elements of all the Mathematicks, improv'd them, and made them more fit for Uſe. Wherefore he is deſervedly to be reckon'd amongſt the chief Compilers of Elements.

Eudoxus of *Cnidos* was not inferior to *Leon*: A Man great in Arithmetick, and to him (if we may believe the *Greek* Scholiaſt) we owe the whole Fifth Book. He likewiſe wrote Elements, and made them more general, and encreas'd the Sections begun by *Plato*; over and above this he was the firſt framer of Aſtronomical Hypotheſes, and derived down the Springs of Geometry, as *Archytas* had done before, to Mechanicks. *Amyclas* the *Heracleot*, and *Menæchmus*, and his Brother *Dinoſtratus*, *Helicon* of *Cyzium*, *Theudius*, *Hermotimus* the *Colophonian*, *Philippus* the *Medmæan*, all Platoniſts, rendered Geometry much more perfect. *Menæchmus* alſo found out the Conick Sections, and by the help of them two mean Proportionals; whoſe Invention in this Caſe is preferr'd by *Eutocius* before any other

other. *Theudius* and *Hermotimus* made the Elements more universal and full. And all these, who were of *Plato*'s Academy, brought Mathematick Philosophy to Perfection, as *Proclus* saith. *Xenocrates* also, one of *Plato*'s Auditors, and Master of *Aristotle*, as well as *Aristotle* himself, were Famous for the Knowledge of the Mathematicks. When a certain Person, who knew nothing of Geometry, was minded to be his Auditor, *Go thy way*, saith he, *for thou wantest the very Handles of Philosophy.*

But of *Aristotle*, what can I say? All his Books are filled with Mathematical Arguments, out of a Collection of which *Blancane* hath made a Book. Two of *Aristotle*'s School are especially celebrated, *Eudemus* and *Theophrastus.* This latter wrote Two Books of Numbers, Four of Geometry, and one of indivisible Lines: The other, compos'd a Mathematical History; and from him *Proclus*, and others have borrowed theirs. To *Aristeus*, *Isidore*, *Hypsicles*, most subtle Geometricians, we are especially indebted for the Books of Solids. Lastly, *Euclid* gathered together the Inventions of others, dispos'd them into Order, improv'd them, and demonstrated them more accurately, and left to us those *Elements*, by which Youth is every where instructed in the Mathematicks. He died in the Year before Christ 284. There follow'd *Euclid* almost an 100 Years afterwards *Eratosthenes* and *Archimedes.* The Name of *Eratosthenes* was very Famous, but his Writings are lost. Many Remains we have of *Archimedes*, and many we have lost.

But

But when I name *Archimedes*, I conceive in my Mind the very Top of human Subtilty, and the Perfection of the whole Mathematical Sciences. His wonderful Inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetzes*, and others. *Conon* was Contemporary to *Archimedes*, one who was both a Geometrician and an Astronomer, whose Death *Archimedes* laments in his Book of the Quadrature of the *Parabola*. There followed *Archimedes* and *Conon*, and that at no great Distance, *Apollonius* of *Perga*, another Prince in Geometry, who was called by way of high Encomium, *The Great Geometrician*. There are extant Four [now Seven] most subtle Books of his Conicks. To the same Person are ascribed, the 14, and 15. Books of *Euclid*, which were contracted by *Hypsicles*. *Hipparchus* and *Menelaus* wrote, this latter 6, the other 12 Books of Subtenses in a Circle; for which Invention, so very profitable and necessary, great Commendations and Thanks are due to both. There are also extent three Books of *Menelaus* concerning Spherical Triangles. Three most useful Books of Sphericks of *Theodosius* the *Tripolite* are also in the Hands of all. And these indeed, if you except *Menelaus*, lived all of them before Christ.

In the Year after Christ 70, there appeared in the World *Claudius Ptolemæus*, the Prince of Astronomers, a Man certainly wonderful, and (as *Pliny* saith) above the Nature of Mortals. But he was not only most skilful in Astronomy, but in Geometry also; which as many other Things written by him do witness, so especially do the Books of Subtenses: Those

of

of *Menelaus* which were Six, and the Twelve of *Hipparchus*, all contracted by him into Five Theorems. As for *Plutarch*, a most fam'd Philosopher, there are extant his Mathematical Problems. And all know of the Learned Commentaries of *Eutocius* the *Ascalonite* upon *Archimedes*. By him are recited the Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sphorus*, *Heron*, as of so many excellent Masters in the Mathematicks, concerning doubling the Cube. *Heron*'s Genius certainly was excellent, as well for Mechanicks as Geometry. The doubling of the Cube delivered by him is commended by *Pappus*, Book 3. *Prop.* 7. before all other. The admirable Works of *Ctesibius* the *Alexandrian*, to whom we owe our Pumps, are celebrated by *Vitruvius*, *Proclus*, *Pliny*, and *Athenæus*. The Name also of *Geminus* is not in the lowest Place amongst Mathematicians, whom *Proclus* has preferr'd in many Things before *Euclid* himself.

Diophantus, and he also an *Alexandrian*, was as great in Arithmetick, as *Archimedes*, *Apollonius*, or *Euclid* in *Geometry*; he was certainly a Master of all Subtilty relating to Numbers; by him was found out that admirable Art, which we call *Algebra*; which in these Times has been rendered more perfect and universal by *Francis Vieta*, and *Renatus Cartesius*. There are others who are celebrated amongst the Ancients also; as *Nicomachus* famous for Arithmetical, Geometrical, and Musical Monuments; *Serenus*, well known to Geometricians for his Two Books, concerning the Section of a Cylinder; *Proclus*,

Pap-

Pappus, *Theon*. How great a Mathematician *Proclus* was, is manifest from his Learned Commentaries on *Euclid*, and other Writings. And this is he, I suppose, who, as *Zonarus* reports, and from him *Ramus*, and *Baronius*, about the Year of Christ 514, with Optic Artifice, and the Glasses which he us d burnt the Fleet of *Vitalian*, who was besieging *Constantinople*. The Praises of *Theon*, which truly are deservedly great, *Peter Ramus* wonderfully exaggerates, insomuch that even the Books which hitherto all have ascribed to *Euclid*, ought, as he thinks, to be attributed to *Theon*. But *Ramus*, who every where is ready to detract from *Euclid*, and this without grounding himself upon any solid Foundation, is not to be hearken'd to here. To come at length to a Conclusion, let *Pappus* bring up the Rear, the last in time amongst the Ancients, as being one who liv'd about the Year 400; but in Reputation, and all Mathematical Commendation to be reckon'd amongst the first. *Alexandria*, that City so fruitful of Great Men, which before had brought forth *Hypsicles*, *Ctesibius* and *Diophantus*, produced him also to the great Advantage of the Mathematicks. He wrote Seven Books of Mathematical Collections, of which the Two First are lost. The Five other do abound with so many, and such various most noble Inventions in almost all Parts of the Mathematicks, that they are esteemed amongst the Chief Monuments of the Ancients which are extant.

And thus you have a short History of the Origin and Progress of the Mathematicks.

From

From which appears the Antiquity, Excellency, and Dignity of this Science. Certainly the ſame Eminent Perſons in the Commonwealth of Learning, who diſcover'd Phliloſophy, diſcover'd alſo the Mathematicks, like two Siſters born at one Birth; whom if any one would violently ſeparate from each other, he certainly attempts to break off their native Concord, with moſt notable Injury, and as it were Cruelty to both; ſeeing, as it is wont to fall out in the Caſe of Twins, where they are remov'd from one another, in Place or by Death, ſo it will be like to happen here, that Mathematicks being plucked away from her, Philoſophy muſt needs languiſh and pine away.

Dr. BARROW's Words, prefix'd before his *Apollonius*.

God always acts Geometrically.

HOW great a Geometrician art thou, O Lord? For while this Science has no Bounds; while there is for ever room for the Discovery of New Theorems, even by Human Faculties, Thou art acquainted with them all at one View, without any train of Consequences, without any wearisome Application of Demonstrations. In other Arts and Sciences our Understanding is able to do almost nothing; and like the Imagination of Brutes seems only to dream of some uncertain Propositions: Whence it is that in so many Men are almost so many Minds. But in these Geometrical Theorems all Men are agreed: In these the Human Faculties appear to have some real Abilities, and those Great, Wonderful, and Amazing. For those Faculties which seem of almost no force in other Mat-

ters, in this Science appear to be Efficacious, Powerful, and Successful, &c. Thee therefore do I take hence occasion to Love, and Rejoice in, and Admire; and to pant after that Day, with the Earnest Breathings of my Soul, when thou shalt be pleased, out of thy Bounty, out of thy Immense and Sacred Benignity, to grant me the favour to perceive, and that with a pure Mind, and clear Vision, not only these Truths, but those also which are more numerous, and more important; and all this without that continual and painful Application of the Imagination which we discover these withal, &c.

Mathematical Notes or Abbreviations.

= The Note for Equality. So $a=b$. signifies that a and b are equal.

+ The Note for Addition. So $a+b$ signifies the Sum of a and b together.

— The Note for Substraction. So $a-b$ signifies the difference between a and b.

× The Note for Multiplication. So $a \times b$ or ab signifies a multiplied by b.

:: The Note for equality of Proposition. So $A : B :: a : b$. signifies that A bears the same proportion to B, that a bears to b.

∺ The Note of continued Proportion. So $A. B. C$ ∺ signifies that A bears the same proportion to B, that B bears to C.

q The Note for a Square. So $C B. q.$ signifies the Square of the Line $C B$.

c The Note for a Cube. So $C B. c.$ signifies the Cube of the Line $C B$.

THE

THE Elements of *Euclid*.

BOOK I.

DEFINITIONS.

1. A Point is a Mark in Magnitude, which is [supposed to be] indivisible.

That is, which cannot be divided so much as in Thought. A Point is the beginning, as it were, of all Magnitude, as Unity is of Number.

2. A Line is a Magnitude which hath Length only, and wants all Breadth; forasmuch as it is understood to be produced from the flowing of a Point.

3. Points are the Terms of a Line.

4. A right Line, is that which lies evenly betwixt its Terms. *Fig. 1. Table 1.*

Or as *Archimedes:* A right Line is the least of all those which have the same Terms; or, is the shortest of all those which can be drawn betwixt two Points.

Or as *Plato* hath it: A right Line is that whose Extremes hide all the rest: [That is, when the Eye is placed in a Continuation of the Line.]

The Sense is the same in all. The Instrument whereby right Lines are described, is [called] a Rule; which whether it be strait or not you may know by this Tryal.

Describe a Line according to the Rule; then turning the Rule so that that which before was the Right-hand End may now become the Left-hand End, apply it again to the Line before described; if it doth now entirely

fall in with the Line, the Rule is ſtrait, if not, the Rule is not ſtrait. The Reaſon hereof depends on Axiom 13.

5. A Surface is a Magnitude which hath only Length and Breadth.

It hath two Dimenſions therefore: And is underſtood to be produced by the flowing of a Line.

6. Lines are the Extremes of a Surface.

7. A Plane, or a plain Surface, is that which lies evenly betwixt its extreme Lines.

Or as *Hero*, that, to all the Parts whereof a right Line may be accommodated.

For it is produced from the Motion of a right Line.

Or, A plain Surface is that whoſe Extremes any of them hide all the reſt, [the Eye being placed in a Continuation of the Surface.]

Or, It is the leaſt of all Surfaces which have the ſame Terms. The Senſe is the ſame in all.

Euclid hath not here defined a Body or Solid, becauſe he was not yet about to treat concerning it. But leſt any one ſhould want the Definition thereof, take it here thus, A Body is a Magnitude long, broad, and deep. A Body therefore hath three Dimenſions, a Surface two, a Line one, a Point none.

8. A plain Angle is the mutual Inclination to each other of two Lines, which touch one another in a Plain, and ſo as not to make one Line.

Fig. 2, 4. Therefore the two Lines B A, C A touching one another in A, but ſo as not to make one Line, conſtitute an Angle.

9. The Sides or Legs of an Angle are the Lines which make the Angle.

10. The *Vertex* or Top of an Angle is the Point (A) in which the Legs do meet and touch one another.

Note, That a ſingle Angle is deſigned by one Letter put at the Top: When there are more at one Point, they are deſigned by three Letters, the middlemoſt of which denotes the Top of the Angle; and many times alſo by one Letter interpos'd betwixt the Sides near the Top. So in Fig. 5. the Angle made by the Lines B A, C A is deſigned either by three Letters B A C, or by one only O.

11. Angles are called Equal, if when the Tops of them are laid upon one another, the Sides of one agree with the Sides of the other. But unto this it is not required that the Sides ſhould be of an equal Length.

12. They

12. They are called Unequal when the Top and one Side agreeing, the other doth not agree; and that is called the Greater, whose Side falls without. So the Angle B A E is greater than the Angle B A C. *Fig. 5.*

An Angle is not diminish'd or increas'd by the Diminution or Augmentation of the Sides that include it.

13. A Right-lin'd Angle is that which right Lines constitute; a curvi-lineal, which crooked Lines; a mixt one, that which a right Line and a crooked one make. *Fig. 2, 4.*

14. When the right Line [C A] standing upon the Right one [B F] leans unto neither Part, and therefore makes the Angles on both Sides equal, C A B = C A F both of the equal Angles are called Right ones: But the right Line C A which stands upon the other, is called a perpendicular Line, or barely a Perpendicular. *Fig. 6.*

A Right Angle may also be defined thus. *Fig. 6.*

A Right Angle is that that (B A C) when on the other Side an equal one ariseth (C A F) if you produce or draw forth a Side, as (BA).

Two Rules so joined as as to contain a Right Angle, make an Instrument, which is called a Square. *Pythagoras* was the Inventor of it, as *Vitruvius* affirmeth, *c.* 2. *l.* 9. So great is the Use and Force of a right Angle in Framing, Measuring, and Strengthning all Things, that nothing almost can be done without it. The Proof of a Square is made thus: Apply the side of it, A E to the right Line A F, and describe the right Line C A along the other Side. Then turning the Square towards B, if on both Sides it agrees to the right Lines C A, A B you may know that it is true and exact. The Reason hereof appears from the 14th Definition it self.

15. The Angle B A C which is greater than the right one F A C is called an Obtuse Angle. *Fig. 7.*

16. The Angle (L A I) which is less than the Right Angle (F A I) is called an Acute one. *Fig. 8.*

17. A plain Figure is a plain Surface, bounded on every Side with one or more Lines.

18. A Circle is a plain Surface contained within the Compass of one Line called the Circumference; from which Line all the right Lines that can be drawn unto one certain Point, within the contained Space (A) are equal. *Fig. 9.*

19. That Point is called the Center.

Fig. 9. 20. The Diameter is a right Line (B A) drawn through the Center, and on both Sides ended at the Circumference; and consequently it divides the Circle into two equal Parts. (As is abundantly manifest from the exact Agreement of two Semicircles when laid one upon another).

21. The Semi-diameter or *Radius* is the right Line A F drawn from the Center to the Circumference.

22. A Semi-circle is a Figure (B L C) which is contain'd by the Diameter B C, and half the Circumference (B L C.)

Mathematicians are wont to divide the Circumference into 360 equal Parts (which they call Degrees,) the Semi-circumference into 180, the Quadrant or Quarter into 90.

23. A Right-lin'd Figure is a plain Surface bounded on every Side with right Lines.

Fig. 10. 24. A Triangle is a plain Surface contained by Three Right Lines.

This is the first and most simple of all Right-lin'd Figures, and that into which they are all resolv'd.

Fig. 10. 25. An Equilateral Triangle is that which hath all the Sides equal.

Fig. 11, 12. 26. An *Isosceles* or equicrural Triangle is that which hath only Two Sides equal.

Fig. 13. 27. A *Scalenum* is that which hath Three unequal Sides.

Fig. 13. 28. A Right-angled Triangle is that which hath One Angle right.

Fig. 12. 29. An Obtuse-angled Triangle is that which hath One Obtuse Angle.

Fig. 10, 11. 30. An Acute-angled Triangle is that which hath Three Acute Angles.

Fig. 14. 15. 31. Amongst quadrilateral Figures, the Rect-angle is that which hath Four right, and consequently equal Angles; whether the Sides be equal or not.

Fig. 15. 32. A Square is that which hath equal Sides, and is Right-angled, and consequently Equi-angled.

Every Square is a Rectangle; but every Rectangle is not a Square.

Fig. 16. 33. A Rhombus is a quadrilateral or four-sided Figure, which is equilateral, but not equiangled.

Fig. 17. 34. A Rhomboides is that which hath the opposite Sides and Angles equal, but is neither Equilateral, nor Equiangled.

35. A

35. A Parallelogram is a quadrilateral Figure, which hath each Two of its opposite Sides (A B, F C and B F, A C) parallel to each other. Now what parallel Lines are will be shewed in the following Definition. *Fig. 12, 15, 16, 17.*

Every Rectangle and Square is a Parallelogram; but every Parallelogram is not a Rectangle or a Square.

36. Right Lines are Parallel or Equi-distant, which being in the same Plain, and drawn out on both Sides infinitely, are distant from one another by Equal Intervals. *Fig. 18.*

The Intervals are said to be equal, in respect of the Perpendiculars. Wherefore if all the Perpendiculars (Q L) unto one of the two Parallels (A B) shall be equal, the right Lines (A B, C F) are said to be Parallel.

Parallels are produced, if the right Line (L Q) which is perpendicular to the right Line (A B) be moved along (A B) always perpendicularly; for then its Extremity L describes the Parallel C F.

37. The Diameter or Diagonal of a Parallelogram, and every Quadrilateral, is a Right Line (A F) drawn through the opposite Angles. *Fig. 17.*

38. Plain Figures contain'd by more Sides than Four, are called Many-sided or Many-angled, and by a *Greek* Word, *Polygones.*

39. The External Angle of a Right-lin'd Figure, is that which ariseth without the Figure when the Side is produced. Such are F B C, G C A, H A B. Every Figure therefore hath so many external Angles, as it hath Sides, and internal Angles. *Fig. 19.*

Postulates.

A Postulate is that which it is manifest in it self, that it may easily be done, or conceiv'd to be done. It is required therefore to be granted that we may,

1. From any Point given draw a right Line unto any other Point given.
2. Draw forth a finite right Line in Length still farther.
3. From any Center at any interval describe a Circle.

Axioms.

Axioms.

AN Axiom is a Truth manifest of it self.

1. Those things which are equal to the same thing, are equal also amongst themselves. And that which is greater or lesser than one of the Equals, is also greater or less than the other of them.

2. If to Equals you add Equals, the Wholes will be equal.

3. If from Equals you take away Equals, the Remainders will be equal.

4. If to Unequals you add Equals, the Wholes will be unequal.

5. If from Unequals you take away Equals, the Remainders will be unequal.

6. What things are each of them half of the same Quantity, are equal amongst themselves; and what things are double, or treble, or quadruple of the same, are equal amongst themselves.

7. What things do mutually agree with one another are Equal.

8. If right Lines be equal, they will mutually agree with one another; and the same thing is true of Angles.

9. The whole is greater than its part.

10. All right Angles are equal amongst themselves.

11. Parallel Lines have a common Perpendicular: That is, the right Line which is perpendicular to one of them is perpendicular also to the other.

Fig. 21. 12. The two perpendicular Lines (LO, QI) intercept equal Parts of the Parallels.

13. Two right Lines do not comprehend a Space; for unto this there are required three at the least.

14. Two right Lines cannot have one common Segment; for that they cut one another only in a Point.

Of Propositions some propose something to be done and are called Problems; in others we proceed no further than bare Contemplation, which therefore are named Theorems.

PROPO-

PROPOSITIONS.

THE Requisite Citations are found in the Margin. When Propositions are cited, the first Number designs the Proposition; the Letter *l* with the Number following, signifies the Book. As when you meet with (*per* 5. *l.* 3.) you must read it thus, (by the 5th Proposition of the 3d Book). The Figure is always to be sought amongst the Figures of that Book in which we are then conversant. The rest of the Citations are easy to be understood.

The primary Affections of Triangles and Parallelograms are deliver'd in this first Book. The more famous Propositions are, 32, 35, 37, 41, 44, 45, 47.

PROPOSITION I. Problem.

UPon a given Right Line (A B) to make an Equilateral Triangle. Fig. 23.

From the Centre A, with the Interval (A B) (*a*) describe the Circle F C B: and from the Centre B with the same Interval B A describe the Circle A C L, cutting the former in the Point C, from which Point draw the Right Lines C A, C B. (a) *Per Postul.* 3.

I say, that the Triangle A C B now made, is Equilateral. For the Right Line A C is equal to the Right (*b*) Line A B, seeing they are Semi-diameters of the same Circle F C B: And again, the Right Line B C is equal to the same Right Line B A, seeing they are both Semi-diameters of the Circle L C A. Therefore A C, B C are (*c*) equal betwixt themselves. And therefore all the Sides of the Triangle are equal. Therefore the Triangle (*d*) A C B is both Equilateral, and made upon the given Line A B; which was the thing to be done. *Q. E. F.* (b) *Per Def.* 18. (c) *Per Axiom* 1. (d) *Per Def.* 25.

Corollary. *Hence we may measure an inaccessible Line, as A B. For suppose any Equilateral Triangle whatsoever B D E applied to the Point B along the Line B A. Looking from the Point B along the Line B E, mark as many Points as you conveniently can in the Line B C. Then remove the* Fig. 77.

Triangle

Triangle BDE along the Line BC, from one place to another of that Line, until by taking aim along the side of the Triangle ED or CF, you see the inaccessible Point A in a Continuation of that Line. Thus the Triangle BAC is as well Equilateral as BDE. If therefore you shall now measure the accessible Line BC, you have the Measure of the inaccessible AB. Q. E. I.

PROP. II. Problem.

Fig. 24. *FRom a given Point A to draw a Right Line equal to one given EF.*

Take with a Pair of Compasses the Interval EF and transfer it from A to D, the Right Line AD will be equal to the given EF.

PROP. III. Problem.

Fig. 24. *TWO unequal Right Lines being given, from the greater of them GH to cut off GI equal to the less EF.*

Take with a Pair of Compasses the Interval of the lesser given Line EF, and transfer it unto the greater from G to I.

PROP. IV. Theorem.

Fig. 25. *IF in two Triangles (X, Z) one side of the one (BA) be equal to one side FL of the other, and another side (CA) of the one equal to another side (IL) of the other, and the Angles (A and L) made by those sides be also equal; then the Bases (BC, FI) are likewise equal, as also the Angles at the Bases (B, F, and C, I) which are opposite to equal sides, and consequently the whole Triangles are equal.*

For

For if we suppose the Triangle Z to be laid upon the Triangle X, the Sides L F, L I will perfectly agree and fall in together with the Sides that are equal to them, A B, A C, and this in such sort (c) that the three Points (L, F, I) shall fall upon the three Points, (A, B, C). Therefore the whole Base F I will also fall upon the whole Base B C. But then the Angles F, B, and likewise those I, C and the [whole Triangles will mutually (*congruere*) agree to each other. All therefore by Axiom 7th. are equal. *Q. E. D.* Which was the Thing to be demonstrated. (c) Per. axi. 8.

Coroll. (1.) *Hence we may also in another way measure the inaccessible Line A B. For from any Point whatsoever, as the Point C, let the Angle A C B be observed, and then let the lines A C, B C be measured: and in any accessible Plain let there be measured about the Angle F, which is equal to the Angle C, Two Lines F D and F E which are equal to the Lines A C and B C respectively. And then there will be the accessible Line D E equal to the inaccessible A B. Q. E. I.* Fig. 78.

Coroll. (2.) *Hence also, those who play at Billiards with Ivory Balls may learn how by the reflexion of their own to hit and remove their Adversaries Ball. For let B be the Ball to be stricken, A that which is to strike it, and C D the Rectilinear Plain. Let the Line B E be perpendicular to the Line C D, and D E be equal to D B. If the Ball A be stricken and carried along the right Line A F E unto the Point F, it will there be so reflected that after the Reflection it will tend unto B. For in the Triangles B F D, E F D, the Side F D is common to both, and the Side D B is equal to the Side D E; and the Angles at D are equal as being right ones. The whole Triangles therefore are equal: and therefore the Angle B F D, which is equal to the Angle D F E, is * equal to A F C, the Angle A F C being vertically opposite to D F E. Wherefore, seeing the Angle A F C is the Angle of Incidence, which in such cases is equal to the Angle of Reflexion], it is manifest that B F D, which hath been proved equal to A F C, is the Angle of the Reflexion of the Ball A, and that the Ball tending towards E is in the Point F so reflected as to hit the Ball B. Q. E. D.* Fig. 79. * Per 15. l. 1.

Scholium,

Scholium or Observation.

BY much the same way of Reasoning, whereby this 4th Proposition has been demonstrated, the following Theorem, which we shall have occasion to use by and by, may be demonstrated also.

Fig. 25. If in Two Triangles X, Z, the Sides B C and F I shall be equal, and the Angles adjacent to these Two Sides equal also, *viz.* B and C equal to F and I: All the other Things, and the whole Triangles themselves will be equal.

For the Side F I laid upon the Side B C will agree, or thoroughly coincide with it (*a*). And then because the Angles B and C are equal to those F and I, when the Side F I is laid upon the Side B C, F L (*b*) will fall exactly upon B A, and I L upon C A. Therefore the Point L will fall upon the Point A (for if it fall without A, the Sides F L I L would not fall upon the Sides B A, C A). Therefore all Things are equal by Axiom 7th.

(a) *Per Axi.* 8.

(b) *Per Axi.* 8.

PROP. V. Theorem.

Fig. 26. *IN an Isosceles or Equicrural Triangle, the Angles at the Base (A, C) are equal.*

Let the Triangle A B C be understood to be twice put, but in an inverted Posture *c b a*. Because therefore, in the Two Triangles A B C, *c b a* the Side A B is by the Supposition equal to the Side *c b*, and the Side C B to the Side *a b*, and the Angle B to the Angle *b*; the Angle A also at the Base will (*c*) be equal to the Angle *c*. Q. E. D. For as for the Angles C and *c* they are the same.

(c) *Per* 4. *l.* 1.

Corollary.

THEREFORE an Equilateral Triangle is also Equiangular.

PROP. VI. Theorem.

Fig. 26. *IF in a Triangle (A B C) Two Angles (A and C) be equal, the Sides also (A B, B C) which are opposite to those Angles are equal also.*

Let

Let the Triangle A B C be supposed to be twice put, but in an Inverse situation, *c b a*. Because therefore in the Triangles A B C, *c b a*, one Side A C is equal to one Side (*c a*,) and the Angle A is equal to the Angle *c*, and the Angle C equal to the Angle *a*, all the other Things shall be likewise (*a*) equal, and consequently A B shall be equal to the Side *c b*. Q. E. D. For as for the Lines C B and *c b* they are the same. (a) Per Schol. Prop. 4.

Coroll.

THEREFORE an Equiangled Eriangle, is also Equilateral.

Coroll. (2.) *Hence, by the means of the Shadow of the Sun, we may measure the Heighth of a Tower, or any elevated Point. For when the Sun is elevated 45 Degrees above the Horizon, the Shadows which the Towers cast towards the Horizon will be exactly equal to their Height. For, by reason that the Angle A C B is half a right Angle, the Angle B A C also* * *will be half a right one; and so, by the force of the present Proposition, the Line A B will be equal to the Line B C. The Line B C therefore being found by measuring, there is found at the same time the Line A B, the Height of the Tower above the Horizon.* Fig. 80. * Per Corol. 11. Prop. 32. l. 1.

Coroll. (3.) *The same Thing also may be found without the Sun by the means of an Astronomical Quadrant. For where the Angle of Elevation is half-right, there the Height of the Tower above the Observer's Eye is equal to the distance of the same Eye, from that Part of the Tower which is opposite to it. The Distance therefore of the Eye from the Tower being given by measuring, there is given at the same time the Height of the Tower. Q. E. I.*

The VIIth Proposition in *Euclid* is for the sake of the VIIIth, which without it will here be demonstrated.

PROP. VIII. Theorem.

IF Two Triangles (X, Z) have all their Sides equal amongst themselves respectively (A C equal to E F; C B to F I; A B to E I;) they will also have all the Angles which are opposite to equal Sides, equal: (C equal to F; A to E; B to I). Fig. 27.

For

For suppose the Side A B laid upon its Equal E I, if then the Point C falls upon F, the Triangles will in the Whole agree or coincide, and consequently all the Angles will be equal. But the Point C will fall upon the Point F. For,

Fig. 81. *From the Centre A let a Circle be described with the Semi-diameter E F; and from the Centre I let another Circle be described with the Semi-diameter I F; the Point C by reason of the Equality of the Sides of both Triangles, will be in the Circumference of both Circles, and consequently in the Point E, the common Intersection of both these Circumferences. Q. E. D.*

PROP. IX. Problem.

Fig. 29. TO *Bisect or Divide into two equal Parts a given right lin'd Angle, as I AL.*

From the Sides of the Angle take with a Pair of Compasses two Equal Lines, A B, A C; then from the Centres B and C describe two equal Circles cutting one another in F; which done draw the Line F A. This bisects the Angle.

For draw the Lines B F, C F; the Triangles F A B F A C are to each other Equilateral; for the Sides A B, A C are by the Construction equal, as in like manner are the Sides B F, C F, they being Semidiameters of equal Circles; and A F is common to both Triangles. Therefore the Angles B A F, C A F (*d*) are equal. Therefore the given Angle I A L is bisected. Q. E. F.

(d) Per 8. l. 1.

Corollary.

HENCE we may learn how an Angle may be divided into equal Angles, 4, 8, 16, *&c. viz.* by bisecting each Part again.

Scholium.

NO one hath hitherto taught the way of dividing Angles into all equal Parts whatsoever with a Pair of Compasses, and a Rule.

Yet may you divide any given Angle Mechanically into any equal Parts whatsoever, if from the Top of the Angle as the

Fig. 30.

the Centre you describe an Arch between the Legs of the Angle, and divide that Arch into as many equal Parts as you require; for right Lines let down from A through the Points of the Division will cut the Angle into so many Equal Parts.

PROP. X. Problem.

TO *bisect a finite given Line (A B).* Fig. 31.

Upon the given A B make an Equilateral (*a*) Triangle A G B. Bisect its Angle G (*b*) with the right Line G C. The same shall bisect the given Line A B.

For in the Triangles X, Z the Side C G is common; and by the Construction G B, G A are equal, and the Angles contained between them A G C, B G C, are likewise equal. Therefore the Bases A C, B C (*c*) are equal. The given Line therefore A B is bisected. Q. E. F.

(a) *Per* 1. *l.* 1.
(b) *Per praeced.*
(c) *Per* 4. *l.* 1.

But for Practice it is sufficient from the Centers A and B to describe two equal Circles, cutting one another in G and L, and so to draw the right Line G L.

PROP. XI. Problem.

FROM *a given Point (A) to raise a Perpendicular in a given right Line (L I.)* Fig. 32.

With a Pair of Compasses take the equal Lines A C, A F. From the Centre C and F describe two Circles, cutting one another in B. The Line which is drawn from B to A will be the perpendicular required.

For let the right Lines C B, F B be drawn. The Triangles X and Z are equilateral to one another. Therefore the Angles C A B, F A B are equal (*a*.) Therefore B A is (*b*) perpendicular to the Line (L I.) Q. E. F.

(a) *Per* 8. *l.* 1.
(b) *Per def.* 14.

In Practice this and the next are easily performed by the help of a Square.

PROP. XII. Problem.

Fig. 33. *FROM a given Point (A) which is without an infinite right Line (as L Q) to let fall a Perpendicular to that Line.*

From the Centre A describe a Circle which may cut the given L Q in C and I. Bisect the right Line C I (c) with the right Line A B. This A B is the Perpendicular required.

(c) Per 10. l. 1.

For let there be drawn A G, A I. Because by the Construction X and Z are equilateral to one another; Therefore the Angles (d) C B A, I B A are equal. Therefore A B is (e) perpendicular. *Q. E. F.*

(d) Per 8. l. 1.
(e) Per Def. 14.

PROP. XIII. Theorem.

Fig. 34. *THE right Line (B A) standing upon the right Line (C F) either makes two right Angles, or Angles equal to two right ones.*

For if B A stand upon it perpendicularly, then by Definition 14 the two Angles B A C, B A F will be right ones. And if B A stand obliquely, let there be rais'd (f) the perpendicular A L. Where because the unequal Angles C A B, F A B possess the same Place which the two right ones C A L, L A F do, and agree to them, they are equal (g) to them. *Q. E. D.*

(f) Per. 11. 1.
(g) Per Axio. 7.

Corollaries.

1. IN the same manner it will be demonstrated, if more right Lines than one stand upon the same right Line, that the Angles thereby made are equal to two right ones.

Fig. 37. 2. Two right Lines cutting one another, make the Angles equal to four right ones.

Fig. 36. 3. All the Angles which are about one Point, make Angles equal to four right ones. It appears from Corollary 2.

4. The

4. The Angle CAF being known, you at the same time know its Complement unto two right Angles BAF. For Example, Let the Angle CAF be of 70 Degrees; the Angle BAF will be of 110 Degrees. For those two Numbers added together make 180 Degrees, which is the Measure of two right Angles. Fig. 37.

PROP. XIV. Theorem.

IF two right Lines (XR, ZR) at the same Point of a right Line QR make the Angles on both Sides (XRQ, ZRQ) equal to two right Angles; the Lines (XR, ZR) make one right Line. Fig. 35.

If you deny it, let XR, BR make one right Line. Therefore the Angles XRQ QRB (*a*) will make two right Angles. Which thing is (*b*) absurd; seeing by the Hypothesis XRQ, ZRQ do make two right Angles.

(a) *Per* 13. 1.
(b) *Contra axio.* 9.

PROP. XV. Theorem.

IF two right Lines (BC, FL) cut one another in A, the Angles opposite at the top (A) are equal, viz. *LAB to CAF, and BAF to LAC.* Fig. 37.

For because BA stands upon the right Line LF, the Angles LAB, FAB are (*c*) equal to two right ones: And because FA stands upon the right Line BC, the Angles FAC, FAB are also equal (*d*) to two right ones. Therefore the two Angles together (*e*) LAB, FAB are equal to those two together CAF, FAB. Therefore taking away the common Angle, FAB, LAB (*f*) remains equal to CAF. In the same manner BAF, LAC are shewed to be equal.

(c) *Per* 13. 1.
(d) By the same Prop.
(e) *Per axio.* 1.
(f) *Per axio.* 3.

Coroll. *From these two Propositions we gather in Catoptrics that a Ray of Light, as reflected in an Angle equal to the Angle of Incidence, taketh the shortest way of all. E. G. When the Angles BED, AEF are equal, the Lines AE and EB taken together, are shorter than any Lines whatsoever, as* Fig. 82.

*AF and FB taken together. For from the Point B let the perpendicular Line BC be let down; and let BD and DC be equal; Let the Lines also EC and FC be drawn. Now in the Triangles BED and DEC, seeing the Side DE is common to both, and the Side BD and DC are equal by the Hypothesis, as is also in the like manner BDE equal to the Angle CDE; the Triangles shall be * equal in all other things, and BE shall be equal to CE, and the Angle BED to the Angle DEC: (where because the Angle DEC is equal to [BED, that is] AEF, the Lines AE, EC are prov'd to make one right Line.) And in the same manner the Line BF will be proved equal to FC. Seeing therefore the Lines BE and EA taken together, are equal to the Line CA, and the Lines BF, FA taken together are equal to the Lines CF, FA taken together; It is manifest that CA, which is one Side of the Triangle ACF†, is less than the two Sides CF, FA taken together.* Q. E. D.

* Per 4. l. 1.

† Per 20. l. 1.

PRÓP. XVI, XVII.

THESE two Propositions *are contain'd in* Proposition 32. *And are not here made use of till then.*

PROP. XVIII. Theorem.

Fig. 38.

IN every Triangle the Angle (A) which is opposed to the greater Side (BO) is the greater; and (B) which is opposite to the lesser Side (AO) is the lesser Angle.

(A) Cannot be equal to (B) for then the opposite Sides BO, AO would be equal (*a*); which is contrary to the Hypothesis. Neither can A be less than B, for if it were so, there might within the Angle B be made an Angle ABF by the right Line BF; which Angle should be equal to A. But then by the 6th of this Book BF, AF shall be equal; and if you add to both OF, then BF, FO shall be equal to AO. But AO by the Hypothesis is less than BO. Therefore BF, FO shall be less than BO, which contradicts the Definition of a right

(a) Per 6. l. 1.

right Line, which is the shortest of all betwixt two Points. Therefore the Angle A is neither less than B, nor equal to it. Therefore it is greater. *Q. E. D.*

PROP. XIX. Theorem.

IN the Triangle AOB the Side (BO) which is opposed to the greater Angle (A) is the greater; And (AO) which is opposed to the lesser Angle B, is the lesser. Fig. 38.

This Proposition is the converse of the former. BO is not less than AO, for if it were, the Angle (A) by the 18th would be less than B; which is contrary to the Hypothesis. Nor can BO be equal to AO, for in this Case by the 5th the Angles A and B would be equal. Fig. 83. But this Equality of those Angles is contrary to the Hypothesis. Therefore BO is greater than AO. *Q. E. D.*

Coroll. *Hence we gather that a Globe or Ball perfectly polished cannot rest in an horizontal plain perfectly polished, but where it touches the Earth. For let the Line AB be an horizontal Plain, C the Earth's Center, CA the Semidiameter of the Earth, perpendicular to the Tangent AB. The Globe placed at B because of its Gravity, and the Declivity of the Plain, will descend towards A. For in the Triangle CAB the perpendicular Line CA, which is opposite to the acute Angle ABC is less than the Line BC which is opposed to the right Angle BAC; and so there is from B to A perpetual Descent, in which the Globe cannot rest. And in the like manner we prove the Descent of Fluids, and their Conformation into a sphorical Surface.*

PROP. XX. Theorem.

IN any Triangle, any two Sides of it taken together, are greater than the remaining Side.

This with *Archimedes* is as it were an Axiom; forasmuch as it is immediately manifest out of his Definition of a right Line; which see above amongst the Definitions.

PROP. XXI. Theorem.

Fig. 39. *IF from the Ends of one Side AB, two right Lines be drawn, and joined together within the Triangle, (as the Lines AO, BO); these are less than the Sides of the Triangle (AC, BC), but they comprehend a greater Angle (AOB.)*

For as for the first Part of the Proposition, Draw out AO unto F. AC, CF are (*a*) greater than AF. Therefore the common Line FB being added, AC, BC are greater than AF, FB. Again, OF, BF are greater (*b*) than OB. Therefore the common AO being added, AF, BF are greater than AO, BO. Therefore AC, CB are much greater than AO, OB.

(a) Per 20. l. 1.

(b) By the same.

The second Part of this Proposition will be demonstrated in the second Corollary of the first Part of Proposition 32. And in the mean while we shall make no use of it.

PROP. XXII. Problem.

Fig. 40. *TO make a Triangle of three given right Lines (BO, LB, LO) (of which any two must be greater than the third.)*

Let BL one of the given Lines be taken, and B one of its extremities being taken for the Centre, with the Interval of the other given Line BO describe an Arch.

Then the other Extremity L being taken for the Centre, with the Interval of the third given Line LO describe an Arch, cutting the former in O; which being done, and the right Lines BO, LO being drawn, I say that that is done which was to be done.

The Demonstration is manifest from the Construction.

PROP.

PROP. XXIII. Problem.

AT a given Point in a right Line (as B) to make an Angle equal to a given one (A). Fig. 40.

First of all let CF be drawn at a venture, cutting the Sides of the given Angle A. Then in the given right Line from B take BL equal to AF. Then from the Centre B describe a Circle with the Interval AC; afterwards another from the Centre L with the Interval FC, which may cut the former in O. Then from O unto B and L having drawn right Lines: The Angle LBO will be equal to the given one A.

For by the Construction the Triangles are Equilateral to one another. Therefore by the 8th of this Book the Angles B and A are equal.

Scholium.

IT seems meet for the Sake of Beginners to propound some things here which are necessary for practice about Angles.

The Measure of an Angle is the Arch of a Circle, which is described from A, the Top of the Angle as the Centre. Therefore look how many Degrees the Arch BC which is intercepted between the Legs of the Angle BAC shall contain, of so many Degrees the Angle BAC shall be said to be. And so because BF a quarter of the Circumference, contains 90 Degrees, and measures the right Angle BAF, a right Angle shall be said to be of 90 Degrees. In like manner because half the Circumference, which is divided into 180 Degrees, measures two right Angles, and the whole Circumference which is divided into 360 Degrees, measures four right Angles; two right Angles shall be said to make 180 Degrees, and four, 360 Degrees. These Things being premised, the Practice about Angles is as follows: Fig. 41.

1. At B a given Point in a right Line to make an Angle equal to the given one A. Fig. 42.

From A the Top of the given Angle as the Centre describe betwixt the Sides the Arch CF. Then from

B the given Point as the Centre describe with the same Interval the Arch LZ; from which take off LO equal to CF. Through B and O draw a right Line; LBO shall be equal to the given A.

Fig. 43. 2. To examine the Degrees of the given Angle OPQ. This is done very easily through any Semicircle or Protractor, which is divided into 180 Degrees. For put the Centre of the Semicircle upon P the Top of the Angle, and the Radius of the Semicircle PL upon the Side of the Angle PQ; and the Arch LO which is intercepted betwixt the Legs of the Angle will shew of how many Degrees the given Angle is.

3. To frame an Ang e containing a given Number of Degrees, as 42:

Fig. 43. Draw the right Line XQ. in which mark the Point P. Upon P put the Centre of the Semicircle, and its Semidiameter PL upon PQ. From L number 42 Degrees, that is, until you come to O. A right Line drawn from P through O, will give the Angle OPL of 42 Degrees.

PROP. XXIV, and XXV. Theorems.

Fig. 44. *IF two Triangles (BAC, BAF) shall have two Sides (BA, AC) equal to two (BA, AF) one side of one, to one side of the other; and if one of the Triangles hath the Angle (BAF) contained by those Sides greater than the other (BAC); it shall have the Base BF greater than the Base (BC.)*

And again, If it hath the Base greater, it shall have the Angle greater.

From the Centre A describe a Circle which passeth through C, it shall pass also through F, because AC, AF are supposed to be equal. Therefore BF shall fall betwixt the Point A and C. Then join CF. The Angle BCF is greater than the Angle ACF; that is, by the 5th of this Book, than the Angle AFC, and consequently much greater than the Angle BFC. Therefore in the Triangle BCF, (a) BF which is opposite

(a) Per 19. l. 1.

to

to the greater Angle BCF is greater than BC which is opposite to the lesser Angle BFC.

2. As for the second Part of the Proposition this is manifest from the first Part.

PROP. XXVI. Theorem.

IF two Triangles (X and Z) have two Angles equal to two, one Angle of the one equal to one Angle of the other (B to F and C to I), and one Side of one Equal to one of the other, whether it be that which is betwixt the equal Angles (as BC=FI) or a Side which is opposed to one of the equal Angles (as AC=LI); all the other Parts shall be equal. Fig. 25.

For in the first Place let the Sides (BC, FI) which are betwixt the equal Angles be supposed equal; In this Case all the other Parts are equal; as hath been already demonstrated in the *Scholium* of Proposition 4.

Again, suppose the Sides AC, LI which are opposed to the equal Angles to be equal. Here because the Angles (B, C) are by the Hypothesis equal, to (F, I) the other Angles also (A, L) shall be equal by Coroll. 9. Prop. 32. which Proposition depends not upon this. Therefore by the first Part of this all the other Parts are equal.

Coroll. *Hence also, following* Thales, *we may measure inaccessible Distances. E. G. Let AD be an inaccessible Line: to which at the Point A let there be erected the Perpendicular AC. Let there be made the Angle (ACB) equal to the Angle (ACD) the accessible Line AB shall be equal to the inaccessible AD. Q. E. I.* Fig. 84.

PROP. XXVII. Theorem.

IF the right Line GO shall cut two right Lines which are parallel (AB, CF); 1. The alternate Angles (RLO, QOL, likewise BLO COL) shall be equal. 2. The external Angle Fig. 45.

GLB

GLB shall be equal to the internal one on the same Side (that is, to LOF:) as likewise GLR equal to LOC. 3. The two internal ones on the same Side (ALO, COL) as taken together, shall be equal to two right ones, as likewise the two (BLO FOL) equal to two right ones.

Fig. 46. The first Part is thus proved. From O and L draw the Perpendiculars OR, LQ. These are perpendicular to the * two Parallels AB, CF; and by Definition 36. equal betwixt themselves, they shall therefore (a) intercept equal Parts of the Parallels, and RL shall be equal to QO. Therefore the Triangles X and Z are equilateral to one another. Therefore (b) the alternate Angles RLO, QOL which are opposite to the equal Sides RO, QL are equal. Which is the first thing. From whence it is likewise manifest that the alternates BLO, COL are equal. For because as well BLO, ALO as COL, FOL are (c) equal to two right ones; therefore BLO, ALO together, are equal to COL, FOL. Therefore taking away the Equals RLO, FOL, the remaining ones BLO, COL shall be likewise equal.

* Per ax. 11.

(a) Per axio. 12.

(b) Per 8. l. 1.

(c) Per 13. l. 1.

Part second. The Angle GLB is equal to that which is vertically opposite RLO (d); But RLO by the first Part of this Proposition is equal to LOF. Therefore GLB the external Angle is equal to the internal remote one which is on the same Side, LOF.

(d) Per 15. l. 1.

Part third. ALO by the first Part is equal to LOF. But LOF with COL make Angles equal to two right ones. Therefore ALO with COL doth the same.

Coroll. *Hence in Imitation of* Eratosthenes *we learn to measure the Compass of the Earth. For he observed that on the Day of the Summer Solstice, the Sun was perpendicularly over* Siene, *a City of* Egypt; *and he found by the means of a Stile perpendicularly erected, that on the same Day the Sun was distant from the vertical Point of* Alexandria *a City of* Egypt, *situate almost under the same Meridian with the other, seven Degrees, with one 5th Part of a Degree; and he knew that these Two Cities were about 5000 Furlongs distant from each other. From these Things by the Help of this Proposition he determin'd the Compass of the Earth. Let A be* Siene, *and B be* Alexandria, *where the Gnomon BC is erected perpendicular to the Horizon. Let DF and EG be the*

Fig. 85.

Solar

Solar Ray's parallel to one another as to Sense. DA a Ray perpendicular to the Horizon of Siene; *and EG a Ray Oblique to the Horizon of* Alexandria, *and which passing by the Top of the Gnomon makes with it the Angle GCF which is of* $7\frac{1}{3}$ *Degrees. Now seeing the Angle GCF is equal to the alternate one AFB, and the measure of it is the Arch AB of* $7\frac{1}{3}$ *Degrees; he found the Compass of the Earth by this Analogy; as* $7\frac{1}{3}$ *Degrees are to* 5000 *Furlongs; so the whole Circumference, which is of* 360 *Degrees, is in a gross Number to* 250000, *the Compass of the Earth in the same Measure. Q. E. I.*

PROP. XXVIII. Theorem.

Fig. 47.

IF a right Line (GO) cutting two right Lines (AB, CF) makes the alternate Angles (ALO, LOF,) equal; the Lines (AB, FC) are parallel.

If you deny it, let XLZ passing through the Point L be parallel to CF. Therefore XLO (*a*) is equal to the alternate FOL, which cannot be, seeing by the Hypothesis ALO is equal to FOL.

(a) By the foregoing.

PROP. XXIX. Theorem.

Fig. 45, & 46.

IF a right Line GO cutting two right Lines (AB, CF) shall make the external Angle (GLB) equal to the internal opposite one (LOF), or shall make the two internal Angles on the same Side (ALO, COL) equal to two right Angles; (AB, CF) are parallel Lines.

By the 15th of this Book GLB is equal to ALO, which is vertically opposite to it. But by the Hypothesis GLB is equal to LOF. Therefore also ALO is equal to its alternate one LOF. Therefore (*b*) AB, CF are parallel.

(b) By the foregoing.

Again, COL with FOL makes Angles equal to two right ones. But by the Hypothesis COL with ALO makes

makes in all two right Angles also. Therefore ALO, FOL the alternate Angles are equal. Therefore, again, (*a*) A B, C F are parallel.

(a) By the foregoing.

Coroll. *From the second Part of this Proposition it appears that every Rectangle is a Parallelogram.*

PROP. XXX. Theorem.

Fig. 45.

IF two right Lines (A B, C F) be parallel to the same right Line (D N) they are parallel betwixt themselves.

It is manifest in it self, and from the foregoing Propositions. For if all be cut by the right Line G O, the external Angle G L B is equal (*b*) to the internal opposite one L D N. Now L D N is an external Angle in respect of D O F, and therefore (*c*) equal to it. Therefore also G L B is equal to L O F. Therefore A B, C F (*d*) are parallel.

(b) *Per* 27. *l.* 1.
(c) By the same.
(d) By the foregoing.

PROP. XXXI. Problem.

Fig. 48.

THrough a given Point (A) to draw a Parallel to a given right Line (C F).

From the Point A let there be drawn at random A L, cutting the given F C. At the Point A let there be made the Angle (*e*) L A S equal to the Angle A L F. The Line A S will be parallel to C F, as is manifest from the 28th, the alternate Angles S A L, A L F being equal.

(e) *Per* 23. *l.* 1.

As for the Practice. Draw AL, and from the Centre L describe an Arch IQ; and from the Center A with the same Interval describe the Arch OX; from which having taken off O B equal to I Q; the right Line drawn through A and B will be the Parallel sought. The Demonstration depends upon the 29th l. 1.

Fig. 49.

Or otherwise thus. From a certain Center P describe a Circle which may pass through the given Point A, and may cut the given Line C F in Q and O. Take the Arch

Arch ON equal to QA. The right Line AN shall be the Parallel sought.

The Demonstration hereof depends upon 29. l. 3, and the 28th of this.

PROP. XXXII. Theorem.

PART I.

IN every Triangle any one of the external Angles (as FBC) is equal to the two internal remote ones (A and C). Fig. 51.

Through the Point B draw (*a*) BL parallel to AC. Because FA cuts the two Parallels BL, AC, the external Angle FBL shall be equal to the internal one A (*b*). And because the Line BC cuts the same Parallels (BL, AC); the Angle LBC shall be (*c*) equal to its alternate one C. Therefore the whole Angle FBC shall be equal to A and C both together. Q. E. D.

(a) *Per* 31. *l.* 1.
(b) *Per* 27. *l.* 1.
(c) By the same.

Corollaries.

1. THE external Angle FBC is greater than either of the internal opposite ones A or C. Fig. 51.

2. Of the Angles (C and AOB) having the same Base, AOB which falls within, is the greater. Fig. 39.

For let AO be produced unto F, AOB by this Proposition is greater than OFB; and likewise OFB is by this greater than C. Therefore AOB is much greater than C.

3. If from one Point A there fall two right Lines upon BC; one of them AO obliquely, the other AF perpendicularly; this last shall fall on the Side of the acute Angle AOB. For let it fall, if it may be, on the Side of the obtuse Angle AOC, as for instance in Q. In this Case the acute Angle AOB shall be external in respect of AQB, and consequently shall be greater than the right one, by Coroll. 1. which is absurd. Fig. 55.

PROP.

PROP. XXXII. Theorem.

PART II.

IN every Triangle the three Angles taken together are equal to two right ones, and therefore make 180 *Degrees.*

Fig. 52. (a) By the first Part of this. (b) *Per* 13. *l.* 1.

Draw forth one Side A B unto F. The external Angle F B C is equal (*a*) to the two internal opposite ones, A and C. But F B C with A B C make (*b*) Angles equal to two right ones. Therefore the two A and C with the same C B A make Angles equal to two right ones. Q. E. D.

Fig. 53. (c) *Per* 27. *l.* 1. (d) *Corol.* 1. *prop.* 13. *l.* 1.

Or thus. Draw the Line H M parallel to A C, the alternate Angles as well O and A, as N and C (*c*) are equal. But O, Q, N make Angles (*d*) equal to two right ones. Therefore also A, C, Q are equal to two right ones. Q. E. D.

Corollaries.

4. THE three Angles of any one Triangle taken toge- are equal to the three Angles of any other Triangle taken together.

5. If in a Triangle one Angle be right (or obtuse) the rest are acute.

6. If in a Triangle one Angle be right, the two other Angles together make one right Angle.

7. In every Triangle, the Angle which is right, is equal to the other two taken together.

8. When you know of how many Degrees one Angle of a Triangle is, you know at the same time how many Degrees the two other Angles as taken together do make up. And so on the contrary, when you know how many Degrees two Angles of a Triangle taken together do make up, or what is the Sum of them, you know at the same time of how many Degrees the third Angle is.

9. When two Angles of one Triangle either severally or together are equal to two Angles of another Triangle, the third Angle of one Triangle is also equal to the third of the other.

10. When

10. When two Triangles have one equal Angle, the Sums also of the rest of the Angles are equal.

11. When in an *Isosceles*, the Angle contained by the equal Sides is a right one, the two others are each of them half-right Angles. And the Angles of an *Isosceles* which are at the Base are always acute.

12. In an equilateral Triangle, each Angle is two thirds of a right Angle. For it is one third of two right ones, therefore it is two thirds of one right one.

13. Hence a right Angle (BAC) is easily divided into three equal Parts; if upon AC be made the equilateral Triangle Z; for seeing FAC is two thirds of one right one, BAF shall be one third of a right one. *Fig.* 54

14. The Perpendicular AF is the shortest of all Lines which can be drawn from the Point (A) unto some right Line. For seeing the Angle F is a right one, AOF shall by Corollary the 5th be an acute one. Therefore (*a*) AF is shorter than any other, as AO. *Fig.* 55. (a) *Per* 19. *l.* 1.

15. Only one Perpeudicular can fall from one Point unto one right Line. This is manifest out of the foregoing Corollary.

16. *Hence also we learn to determine the Parallax of the Stars, or the Difference of their true and apparent Place. Let A be the Centre of the Earth, B the Place of the Observer upon the Surface of the same. Let DBC be the Angle of the Star C according to Observation, or the* visible *Angular Distance thereof from the vertical Point; when in the mean while DAC is the* true *angular Distance. Now the external Angle DBG which is given from Observation is equal to the Angles BAC and BCA taken together; and consequently the Angle BCA is the Difference of the Angles DBC and DAC. If therefore we shall from Astronomical Tables seek the Angle DAC, or what at the time of Observation, is the true angular Distance of the Star from the vertical Point, when the Angle DBC is at the same time known by means of the Quadrant, the Difference of those Angles BCA, which we call the* Parallax, *will likewise be known. Q. E. I.* *Fig.* 86.

Scholium.

BY the Testimony of *Eudemus* an ancient Geometrician, *Pythagoras* was the finder-out of this Proposition,

tion, which indeed is a Theorem most excellent in it self, most fruitful in its Consectaries, and of use in all Parts of the Mathematicks. *Aristotle* very frequently makes mention of it, who also puts it for an Example of the most perfect Demonstration. But like as from this Proposition we have already learned, how many right Angles the Angles of a Triangle are equivalent to; so by the help of the same, it will in the three following Propositions be manifest, how many right Angles, the Angles of any rectilinear Figure whatsoever, whether internal or external do make.

Theorem 1.

Fig. 56. IN every quadrangular Figure the four Angles together make four right ones.

For if through the opposite Angles you draw the right Line B F, this will cut the Triangle into four Quadrangles, without forming any new Angles, whose Angles together do (*a*) make four right Angles.

(a) *Per* 32. *l.* 1.

Theorem 2.

ALL the Angles together of every right-lin'd Figure make twice so many right ones, abating four, as are the Sides of the Figure.

Fig. 57. From any Point A within the Figure let there be drawn unto the Angles of the Figure right Lines, which shall cut the Figure into so many Triangles as it hath Sides, and make no more Angles but those of the Center. Wherefore when each of the Triangles contains two right Angles (*b*), they must all together contain twice so many right Angles as there are Sides. Now the Angles about the Point A, (*c*) do make four right Angles. Therefore if from the Angles of all the Triangles you take away the new Angles which are about A, the remaining Angles which indeed do alone constitute the Angles of the Figure, will make twice so many right Angles, excepting four, as are the Sides of the Figure.

(b) *Per* 32. *l.* 1.

(c) *Corol.* 3. *prop.* 13. *l.* 1.

Hence it appears that all Right-lin'd Figures of the same Species, or Number of Sides & Angles have the Sums of their Angles equal. Which thing is worthy of Admiration.

The

The Practice is thus; Double the Denominator of the Figure; and from the Product take away four; the Remainder is the Number of the right Angles, which the internal Angles of the Figure do make.

Theorem 3.

ALL the external Angles of any right-lin'd Figure whatsoever taken together do make up four right Angles. Fig. 58.

For each of the internal Angles of the Figure does (*d*) with its respective external one make two right Angles. Therefore all the internal ones, together with all the external ones, do make up twice so many right Angles as are the Sides of the Figure. Now by the Precedent, the internal ones, together with four right Angles added to them, make twice so many right Angles as are the Sides of the Figure. Therefore the external Angles are equal to four right ones. (d) *Per* 13. *l.* 1.

Wonderful truly is this Property of right-lin'd Figures; from whence it follows also, that all the right-lin'd Figures of any Species whatsoever have the Sums of their external Angles equal. And therefore the three external Angles of a Triangle are equal to the thousand external Angles of a thousand-sided Figure. Which Observation is altogether worthy of Admiration.

PROP. XXXIII. Theorem.

IF two right Lines, which are equal and parallel, as (AB, CF) be joined by two others (AC, BF); these also will be equal and parallel. Fig. 59.

Let AF cut the Parallels AB, CF. In the Triangles Q, R, the alternate Angles BAF, CFA (*a*) will be equal. Now the Side AB is supposed equal to the Side CF, and AF is common to both Triangles. Therefore (*b*) the Bases BF, AC are equal. (Which is the first Part.) And then the Angles at the Bases AFB, FAC are equal; and therefore AF falling upon the right Lines AC, and BF, makes the alternate Angles AFB, (a) *Per* 27. *l.* 1. (b) *Per* 4. *l.* 1.

(c) *Per* 28. *l.* 1. F A C equal. Therefore A C, B F are also (*c*) parallel. Which is the other Part.

Fig. 87. Coroll. *Hence we learn to measure as well as the Heights of Mountains above the Horizon as their horizontal Lines. Let A B C be the Side of a Mountain, to which apply a great Square, or some Instrument equivalent thereto A D B. Then shall A D be equal to H B, and D B equal to A H. Then coming unto the lower Part which is from the Point B unto the Point C practise as before. So shall E B be equal to C F, and E C be equal to B F. Which done, the Sides parallel to the Horizon, A D, B E, &c. added together will give the horizontal Line G C; and the perpendicular Sides B D, E C, &c. added together will give the Height A G.*

Fig. 59. Coroll. (2.) *Hence also we learn to estimate the Composition of Motions. Let a Body placed at A be driven in the same Moment of Time by the Force A C according to the Direction of the Line A C, and by the Force A B according to the Direction of the Line A B. From the Conjunction of these two Forces it will describe the Diagonal A F. For in this Line of its Motion neither of the Forces is changed. For the Body at F is equally distant from both the Lines of Direction A C, A B, as if it had been driven by either of the Forces separately; which thing can be said of no other Point. And this Corollary doth so fully agree with Astronomical and other Mechanical Phænomena, that it is worthily reckoned by the Famous Sir* Isaac Newton, *as a Foundation of his Geometrical Philosophy.*

PROP. XXXIV. Theorem.

Fig. 59. *IN every Parallelogram the opposite Sides and Angles are equal, and it is cut into two equal Parts by the Diameter.*

Because A B, C F are (*a*) parallel, and A F falls upon them, the alternate Angles B A F, C F A are (*b*) equal. Likewise because A C, B F (*c*) are parallel, and upon them falls the Line A F, the alternates C A F, B F A (*d*) are parallel. Therefore the whole Angle B A C is equal to the whole Angle B F C. In the same manner B and C are shew'd to be equal. Which was the first Part.

(a) *Per defin.* 35.
(b) *Per* 27. *l.* 1.
(c) *Per Def.* 35.
(d) *Per* 27. *l.* 1.

Now

Now becauſe it hath been already ſhew'd that the Triangles Q, R, which have one common Side A F, have alſo the Angles adjacent to the common Side equal, B A F to C F A; and C A F, to B F A; the Sides likewiſe ſhall be equal, A B to F C, and B F to A C; and thus the whole Triangles are equal. Which was the ſecond Part.

Scholium.

PArtly from this Theorem, and partly from a Definition to be premis'd to the ſecond Book, the meaſuring of a right-angled Parallelogram is eaſily deduced. The Area thereof being produced by the Multiplication of the two contiguous Sides A F, A C one by another. *E. G.* Let A F be a Line of 8, A C a Line of 4 Feet. Multiply 8 by 4, there ariſes 32 Square Feet for the Area of the Rectangle. *Fig. 66.*

But the Area of a Square is had from the Multiplication of the Side F I by it ſelf; as if F I be of 5 Feet, multiply 5 into it ſelf, there will ariſe 25 ſquare Feet for the Area of the Square. *Fig. 67.*

The Demonſtration is manifeſt from this Propoſition, if parallel Lines be drawn through the Diviſions of the Sides.

Corollary, *Hence Surveyors do eaſily divide the Area of a Field when it is a Parallelogram. For let A B C D be the parallelogram Field: A D the Diameter or Diagonal Line of the ſame; the middle Point whereof is marked F. Whatſoever right Line, as E G, paſſeth through the Point F, it divides the Field into equal Parts E A C G, E B D G. For the Triangle A B D is equal to the Triangle A C D, and * the Triangle A E F equal to the Triangle G F D. If therefore to the Trapezium E B D F, inſtead of the Triangle A E F, you ſhall add the triangle which is equal to it G F D, you will not change the Area; but the Trapezium E B D G will be equal to the Triangle A B D or to half the Parallelogram, and conſequently equal to the Trapezium A E G C. Q. E. I.* *Fig. 88.* * Per. 26. l. 1.

PROP. XXXV, XXXVI. Theorems.

Fig. 62. *PArallelograms upon the same or equal Bases (AB) and between the same Parallels (CQ, AX) are equal.*

(a) *Per Def.* 35. (b) *Per* 27. *l.* 1. (c) *Per* 34. *l.* 1. (d) *Per* 34. *l.* 1. (e) *Per* 4. *l.* 1.

Because AL, BQ (*a*) are parallel, and CQ cuts them, the external Angle CLA shall (*b*) be equal to the internal one FQB. Then because as well CF as LQ are equal (*c*) to the same AB, CF is equal to LQ. Add then FL to both, the whole Lines CL, FQ are equal. Moreover AL, BQ are equal (*d*). Therefore the Triangles CLA, FQB (*e*) are equal. Therefore taking away the common Triangle FOL, the Plains FOAC, QBOL remain equal: To each of which Trapeziums add the Triangle AOB, the whole Parallelograms ACFB, ALQB become equal. Q. E. D.

This Proposition will be made universal, *Prop.* 1. *l.* 6. Beginners may here observe, that although of two Parallelograms which are between the same Parallels infinitely produced, and upon the same Base, one of them be extended unto an infinite Length, it still remains but equal to the other, by the Force of the present Demonstration.

[*From Hence it follows, that two Cities in Magnitude equal, may so much differ in Compass, that the Circumference of one may exceed that of the other an hundred or a thousand Times. If for instance, one be of a square Figure or Rectangular; but the other a Parallelogram, betwixt the same Parallels indeed with the former, but very oblong.*

Moreover it hence follows, that Figures of equal Compass round may contain Area's vastly different.]

Scholium.

Fig. 62. FROM this Theorem we may learn to measure any Parallelogram. For the Area of it is produced from the perpendicular Altitude QX, or CA multiplied into Base AB.

For the Area of the Rectangle C B which is equal to that of the Parallelogram B L is made (*a*) by A C, multiplying A B. Therefore, *&c.*

(a) By the foregoing Scholium.

PROP. XXXVII, XXXVIII. Theorems.

TRIANGLES (ACB, AFB) upon the same or equal Bases (AB), and between the same Parallels (CI, AZ) are equal. Fig. 63.

Draw the Lines B L, B I parallel to the Sides A C, A F. The Parallelograms A C L B, A F I B (*b*) are equal. But the given Triangles are halves of those Parallelograms (*c*). Therefore the given Triangles (*d*) are equal.

(b) By the foregoing.
(c) *Per* 34. *l.* 1.
(d) *Per axio.* 6.

This Proposition will be made universal, *Prop.* 1. *l.* 1. Let Beginners mark the same Thing here concerning Triangles, which we bid them to note in the foregoing Proposition concerning Parallelograms.

Coroll. (1.) *Hence Surveyors easily divide the Area of a triangular Field. Let ABC be the Field, and let the Base BC be bisected in D. The Triangles ABD, ADC upon the equal Bases BD and DC, and having a common Top A, or being between the same Parallels, are equal.* Q. E. F. Fig. 89.

[Coroll. (2.) *Hence we also gather, with the Famous Sir* Isaac Newton, *that the Area's with all Bodies whatsoever that revolve round about an immoveable Center, towards which they are impell'd, do describe, are both in immoveable Planis, and are proportional to the times of Description. For let the Time be divided into equal Parts, and in the first equal Part of Time, let the Body by the impress'd Force describe the right Line AB. The same Body in the second Part of Time, if nothing hindred, would go forwards strait unto c, describing the Line B c equal to A B; so that the Area's made by Lines drawn from the Center ASB, BSc (a) would be equal. But when the Body comes unto B, let the Force act with one single Impulse, but a great one, and make the Body to deflect from Bc, and to go forwards in the right Line BC; i. e. let the centripetal Force be in that Place to the Force before impuls'd as Cc or Bg is to Bc; in this Case the Body will (b) describe the Diagonal B C. Let there be drawn parallel to BS the right Line Cc meeting BC in C. In the second* Fig. 90.

(a) *Per* 37. *l.* 1.

(b) *Per Corol.* 2. *prop.* 33. *l.* 1.

Part of Time compleated, the Body will be found in the Point C in the same Plain with the first Triangle SAB. Join SC. The Area made by a Ray drawn from the Center, that is the Triangle SBC will be equal to (c) SBc, and consequently to the first Angle SAB (d). By the same Argument the Body in the third equal Part of Time would by its present Force reach from C unto d, so that the Line Cd should be equal to the Line Bc or AB. But if the centripetal Force, whether it be greater or lesser, does again act upon it in the Point C, in the end of the third Part of Time it will be found somewhere in the Line Dd, parallel to SC; and therefore as before, supposing the said Force to be equal or unequal to what it was before, it will be found to have described the Diagonal CD, and will be found in the Point D, and a Ray being drawn from the Center, the Triangle SDC will be equal to that SdC, and consequently to the others SCB, SAB which are equal one to the other. In like manner if the centripetal Force act successively in the Points D, E, F, and be the cause that the Body in the several Parts of time respectively describes the Diagonals, DE, EF, &c. the Area's now made as afore will be in the same Plain, and Triangles will be described equal to the former Triangles. Therefore in equal Times equal Area's are described in an immoveable Plane; and so the Sums of the Area's SADS, SAFS will be amongst themselves as the Times wherein they were described. Now let the Number of the Triangles be increased, and their Wideness diminished infinitely; both that last Perimeter of them ABCDEF, will be a curve Line, and the Area's described in one and the same immoveable Plane will in this Case also be proportional to the Times as well as before. Q. E. D.

(c) Per. 37. l. 1.

(d) Per Axio. 1.

PROP. XXXIX, XL. Theorems.

Fig. 64.

Equal Triangles (ACB, AFB) upon the same or an equal Base (AB) and on the same Side, are between the same Parallels, (AB, CF.)

If you deny it, let CL be parallel to AB and let BL be drawn. Then ALB is equal to ACB (*a*). But by the Hypothesis AFB is equal to ACB. Therefore ALB

(a) By the foregoing.

A L B and A F B are equal; *i. e.* a part is equal to the whole. Which cannot be. Therefore, *&c.*

[Coroll. (1.) *Hence also, with the Famous Sir* Isaac Newton, *we gather, that all Bodies which are moved in curve Lines, and describe Area's about some Center proportional to the Times, are perpetually urged and press'd by a Force impelling towards the Center. For because of the Equality of the Triangles SCB, ScB described upon the same Base SB, the Points C and c shall be in a Line Cc which is parallel to the Base; and so the Figure BcCg shall be a Parallelogram; the Sides whereof Bc and Bg are * the Lines of the Directions of the Forces, and BC is the Diagonal. The Body therefore is urged unto C by the Force Bg which tends unto S the Center. And so in all the Points,* C, D, E, F. Q. E. D.

* *Per Coroll* 2. *prop.* 33. *l.* 1.

Coroll. (2.) *Seeing therefore in the Motion of the primary Planets, the Area's made by Rays, or right Lines drawn from them unto the Sun, are always proportional to the Times, as all Astronomers know, the Planets are urged by a perpetual Force, which tends to the Sun. And the same thing is equally true of the secondary Planets with respect to their primary ones.*]

PROP. XLI. Theorem.

IF a Triangle (A F B) be in the same Parallels with a Parallelogram (A L) and have the same or an equal Base (AB), It is half of the Parallelogram. Fig. 65.

Draw CB. The Triangles A F B, A C B are (*a*) equal. But A C B is half of the Parallelogram A L (*b*). Therefore A F B also is half of A L. Q. E. D.

(a) *Per* 37. 38. *l.* 1.
(b) *Per* 34. *l.* 1.

Scholium.

FRom this Proposition, with the Scholium of Prop 35. we learn that the Area of whatsoever Triangle, as A F B, is produced from half the Altitude F I multiplied into the Base AB, or half the Base multiplied into the Altitude. Wherefore one Side of a Triangle being known, and the Heighth, that is, the Perpendicular which falls upon the known Side from the opposite Angle, the Measure of the Triangle is given. As if the Base AB Fig. 65.

be of an 100 Feet, the Heigth F I 85. multiply half the Base 50 by 85, and you have the Area of the Triangle A F B=4250 Feet Square. Further, the Altitude of a Triangle, when the Area of it is in all Points accessible, may be known mechanically as well as the Sides. But if the Area of it cannot be gone over, the Heighth may be found Geometrically by 12 and 13. *lib.* 2. as we shall there shew.

In a rectangle Triangle, the Heighth is the same with either of the Sides about the right Angle. Half of this therefore multiplied into the other Side adjacent to the right Angle will give the Area of the Triangle.

PROP. XLII. Problem.

Fig. 66. TO *make a Parallelogram with an Angle equal to a given one* (O); *and equal to a given Triangle* (*A C B.*)

Bisect the Base A B in F. Through C draw C X parallel (*a*) to A B. Make the Angle B A L equal to the given one O (*b*). Draw F I parallel (*c*) to A L. A L I F shall be that which was sought for.

For let F C be drawn. The Parallelogram A I hath an Angle L A F equal to the given One O, and is equal to the given Triangle A C B; seeing that as well the Triangle A C B (*d*) as the Parallelogram A I (*e*) is double to the same Triangle A C F.

(a) *Per* 31. *l.* 1.
(b) *Per* 23. *l.* 1.
(c) *Per* 31. *l.* 1.
(d) *Per* 38. *l.* 1.
(e) By the foregoing.

Corollary.

Fig. 66. THE Triangle A C B being given, a Rectangle equal to it is had, if there be drawn a Line Parallel to the Side A B, and A B being bisected in F, the perpendicular B Q be erected. For the Rectangle under F B and Q B will be equal to the Triangle A C B.

PROP.

PROP. XLIII. Theorem.

IN a Parallelogram (as B L) the Complements (B O, O L) of those Parallelograms which are about the Diameter (R F, C S) are equal. Fig. 67.

If through any Point of the Diameter A Q as the Point O, C F be drawn parallel to the Side A B, and R S parallel to the Side B Q; the whole Parallelogram B L is divided into four Parallelograms, whereof two are about the Diameter R F, C S, the other two B O, O L are the Complements of these unto the whole Parallelogram B L.

Their Equality is thus proved. The Triangles (*f*) A B Q, A L Q are equal. Likewise the Triangles A R O, O C Q (*g*) are equal to the Triangles A F O, O S Q. Therefore if from the Equals A B Q, A L Q you take away Equals, on this Side A R O, O C Q, on that A F O, O S Q; then B O and O L shall remain equal. Q. E. D.

(f) Per 34. l. 1.
(g) By the same.

PROP. XLIV. Problem.

UPON a given right Line (O S) to constitute a Parallelogram, in a given Angle (X,) which Parallelogram shall be equal to a given Triangle (V). Fig. 64.

Make a Parallelogram (*a*) R C equal to the given V, having its Angle R O C equal to the given one X, and join the Side R O directly to the given Line O S, so as to make one right Line therewith. Then through S draw S Q (*b*) parallel to O C, which S Q let B C meet when it is produced unto Q. Then let a right Line drawn through Q and O meet B R produced unto A. Which done, through A draw A L parallel to O S, which A L let C O and Q S meet when it is produced unto F and L; The Parallelogram O L is that which was required.

(a) Per 42. l. 1.
(b) Per 31. l. 1.

For

(c) By the foregoing.

For OL (*c*) is equal to RC, that is, by the Construction, to the given Triangle V, and is at the given Line OS; and (*d*) the Angle FOS is equal to the Angle ROC; that is, by the Construction, equal to the given Angle X.

(d) *Per* 15. *l.* 1.

Scholium. *This Proposition contains a certain Geometrical Division. For in the vulgar Arithmetical Division, the Number to be divided may justly be considered as being a certain Rectangle.* E. G. *Let the Rectangle AB comprehending* 12 *square Feet, be to be divided by* 2; i. e. *a Rectangle is to be found equal to that AB of* 12 *square Feet, one of whose Sides shall be only* 2 *Feet: From whence it comes to be enquired of what Number the Side sought shall consist; which Side is to be esteemed a certain Quotient of this Division. Which Thing is performed Geometrically after this manner. With a Pair of Compasses take the Line BD of two Feet, and draw the Diagonal DEF. The Line AF is that which is sought for. For the Complements EG and EC are* * *equal; and in the Rectangle EG one Side EH is equal to the Line BD which is of* 2 *Feet; and the Side EI is equal to AF.*

Fig. 91.

* *Per* 43. *l.* 1.

This Kind of Division is called Application, *because the rectangular Space AB is* Applied *to the Line BD or EH; and hence it comes that Division is often named* Application; *respect being had to the Practice of the old Geometricians, who always made more Use of Geometrical Construction, which requires only a Rule, and a Pair of Compasses, than of Arithmetical Computation, which is performed by Numbers.*

PROP. XLV. Problem.

Fig. 69.

UPON a given Line (IQ) and in a given Angle (H) to make a Parallelogram equal to a given Rectilineal Figure (CBA).

Resolve the given Rectilinear into the Triangles A,B,C, by drawing the right Lines FL, FI.

Upon the given Line IQ in the given Angle H make (*a*) the Parallelogram IV equal to the Triangle A. Then the right Line IR being produced infinitely towards P; Upon the right Line RV in the Angle VRP (*b*) make the Parallelogram RZ equal to the Triangle B. Again, upon

(a) *Per* 44. *l.* 1.

(b) By the same.

upon the Line S Z with the Angle Z S P make the Parallelogram S G equal to the Triangle C. This done I say I G is the Parallelogram sought for.

For (c) the Angle Z V R is equal to its alternate I R V. But (d) Q V R and I R V are equal to two right Angles. Therefore also Q V R and Z V R are equal to two right ones. Therefore * Q V and Z V fall directly so as to make one right Line. After the same manner I might shew that Q Z and Z G make one right Line. Therefore the whole Q V Z G is one right Line, and is also parallel to I X, seeing by the Construction Q V is parallel to I P. Now X G also (e) is parallel to I Q. Seeing X G is parallel to S Z, and S Z to R V, and R V to I Q.

(c) *Per* 27. *l.* 1.
(d) By the same.
* *Per* 14 *l.* 1.
(e) *Per* 30. *l.* 1.

I G Therefore (f) is a Parallelogram; but that it is such an one as was required, is manifest from the Construction

(f) *Per def.* 35.

[Coroll. *Hence is easily found the Excess whereby a greater rectilinear Figure exceeds a lesser. To wit, if unto the same right Line I Q be applied Parallelograms respectively equal to the two right-lin'd Figures. For that Parallelogram by which the greater Rectilinear exceeds the lesser will give the difference of them.* Q. E. I.]

Scholium.

WE will here add a Problem that will be useful for the Practice of Proposition 14. l. 2.

A quadrangular Figure B F being given, to describe an equal Rectangle. *Fig.* 70.

Resolve it into Triangles by the right Line A C. From the opposite Angles let down the Perpendiculars B O, F I. Bisect A C in S. From S erect the Perpendicular S L, equal to the two B O, F I put together. The Rectangle comprehended under L S and S A is equal to the given B F. The Demonstration appears out of Proposition 41.

PROP.

PROP. XLVI. Problem.

Fig. 71. *FROM a given right Line (A B) to describe a Square.*

Erect two Perpendiculars equal to the given AB; to wit, AC, BE, then join CE. I say the Thing is done. For seeing the two Angles A and B are (g) right ones, AC and BE shall (h) be parallel; but they are also (a) equal. Therefore CE and AB are (b) parallel and equal. Therefore the Figure is a Parallelogram and Equilateral. But all the Angles also are right ones (for seeing A and B are right Angles, the opposite ones (c) E and C are right also. Therefore the Figure AE is a Square.

(g) By the Construction
(h) Per 29. l. 1.
(a) By the Construction
(b) Per 33. l. 1.
(c) Per 34. l. 1.

[*In the same manner you may easily describe a Rectangle which hath the two unequal Sides given.*]

PROP. XLVII. Theorem.

Fig. 72. *IN every right-angled Triangle (as ABC) the Square of the Side (AC) which is opposite to the right Angle is equal to the two Squares together of the two other Sides (AB, CB.)*

Let IC and BF be drawn; and BE parallel to AF. Now if to the right and therefore equal Angles IAB, FAC, there be added the common Angle BAC, the Wholes IAC and FAB shall be equal. But in the Triangles IAC, FAB, the Sides which contain those equal Angles are equal (d) amongst themselves, to wit, IA, CA, to BA, AF, each to each. Therefore the Triangles IAC, FAB (e) are equal. Which because they stand upon the same Bases IA, FA with the Parallelograms ABLI and ZAFE, and between the same Parallels IA, LBC, and AF, EZB, they are halves (f) of those Parallelograms. Therefore the Parallelograms ABLI, ZAFE, as being Doubles of Equals, are equal betwixt themselves. By the same reasoning the right Lines AX, BR being drawn, it may be shewn that

[d] Per def. Square.
[e] Per 4. l. 1.
[f] Per 41. l. 1.

that the Parallelograms EC, BX are equal. Therefore the whole AR is equal to IB and BX together. *Q. E. D.*

It was taken for granted that LBC is parallel to IA, in order to which LB and BC must be one right Line. Now that they are so is manifest from the 14th, seeing the Angles LBA and CBA are both right ones, by the Hypothesis.

Scholium.

THIS Theorem (which *Prop.* 31. *l.* 6. *Euclid* extends unto all like or similar Figures) is commonly call'd the *Pythagoric* Theorem, from *Pythagoras* the Inventor of it; who, as is attested by *Proclus*, *Vitruvius* and others, offer'd Sacrifices to the Muses, as supposing himself to have been helped by them in so excellent an Invention; in which thing he shew'd himself to be ignorant of God, the Lord of Sciences, the true and only Author of all Wisdom; or certainly if he knew him, he glorified him not as God. There is frequent and notable Use of this Theorem through all theMathematicks; and in particular it opens a way unto the Knowledge of incommensurable Magnitudes, a main Secret of Geometrical Philosophy.

That the Side of a Square is incommensurable to the Diameter is a thing much celebrated amongst the old Philosophers, *Aristotle* and *Plato* especially; insomuch that *Plato* would say, that he who knows not this is not a Man, but a Beast. Now the Knowledge of this Mystery seems to have taken its Rise out of this 47th Proposition. For seeing in the Square AE the Angle A is a right Angle, the Square of the Diameter CB shall be equal to both the Squares of the Sides, AB, AC, and therefore double to one of them. Wherefore seeing the Square of CB is 2, and the Square of the Side AB is 1 or Unity, the Diameter CB shall be the square Root of 2, and the Side AB the square Root of Unity, or Unity it self; the *Ratio* of which Quantities (as it will be demonstrated in its Place) cannot be explicated in Numbers, and therefore they are incommensurable. *Fig.* 71.

And by this one Argument alone, if all others were wanting, it might evidently be made out, that Geometrical Magnitudes cannot be made up of a definite Number

ber of Points; for otherwise none would be incommensurable; forasmuch as a Point would be the common Measure of all.

To these Things we will subjoin three Problems which are deduced out of the present Proposition, and are of frequent Use.

Problem 1.

Fig. 73. IF any Number of Squares are given, to make one equal to them all together.

Let there be three or more Squares given, whose Sides are A B, B C, C E. Make the right Angle F B Z having indefinite Sides, and unto the Sides of it transfer A B and B C, and then join A C. The Square of A C shall be equal to the Squares of A B, and B C together (*a*). Then transfer A C from B unto X, and C E the third given Side transfer from B nnto E, and join E X; the Square of E X shall be equal to the Squares of E B (or E C) and B X together; that is, equal to the three given Squares, whose Sides are A B, B C, C E: And so on as long as you please.

[a] Per 47. l. 1.

Problem 2.

Fig. 74. TWO unequal right Lines being given (A B, B C) to determine that Square, whereby the Square of the greater (A B) exceeds the Square of the less (B C.)

From the Center B with the Interval A B describe a Circle. Then from C erect a Perpendicular C E, cutting the Circumference in E. The Square of C E is the Excess or Difference which is sought for.

For let B E be drawn. The Square of B E, that is, of A B, is equal to the Squares (*a*) of B C and C E together. Therefore, *&c.*

[a] Per 47. l. 1.

Problem 3.

Fig. 75. ANY two Sides of a right-angled Triangle being known, to find the third.

Let the Sides containing the right Angle be A B, A C, the

the one of 6 Feet, the other of 8. You are to find of how many Feet the Side CB, which is opposite to the right Angle is. To do which, multiply 6 and 8 each of them by it self. From which Multiplication there will arise for the Squares of those two Sides 36 and 64; the Sum of which is an 100. The square Root of an 100 which is 10, gives the Feet of the Side BC, whose Quantity was sought. This Demonstration offers it self in and from this 47th Proposition, for the Sum of the Squares BA and CA is equal to the Square of BC. Therefore the Root of the Sum of them is equal to the Root or Side BC.

Then let the Sides AB, BC be known, the one of 6 Feet, the other of 10, you are now to find AC. Take the Square of the Side AB which is 36, out of the Square of the Side BC=100. The Remainder 64 shall be the Square of the Side AC. The Root therefore of 64, which is 8, gives the Feet of the Side AC.

Corollary, *From hence we derive the Original of the Tables of Sines, Tangents, and Secants. For, for Instance, let AC the Semidiameter of the Circle be of 100,000 Parts, and the Angle BAD of 30 Degrees. Because the Chord or Subtense of 60 Degrees * is equal to AC the Semidiameter; BD the Sine of 30 Degrees shall be equal to half the Semidiameter, or ½ AC; and therefore shall contain 50,000 Parts. But now in the right-angled Triangle ADB, the Square of AB is equal to the Squares of AD and BD. Wherefore let the Semidiameter AB be squared (by multiplying 100,000 by 100,000) and from that Square substract the Square of BD. The Remainder shall be the Square of AD, or of the Cosine equal to it BF; out of which extract the square Root, and you will have the Line BF or AD. Then by this following Analogy, AD : BD :: AE : CE, or AD : BD :: AC : CE, will be had the Tangent CE. And then lastly, if the Square of AC be added to the Square of CE, the Root of the Sum being extracted will be the Secant AE.* Q. E. I.

Fig. 92.

* *Per Corol. 1. prop. 15. l. 4. and Corol. 2. prop. 3. l. 3.*

PROP. XLVIII. Theorem.

IF in a Triangle the Square of one of the Sides (AB) be equal to the two Squares of the other Sides

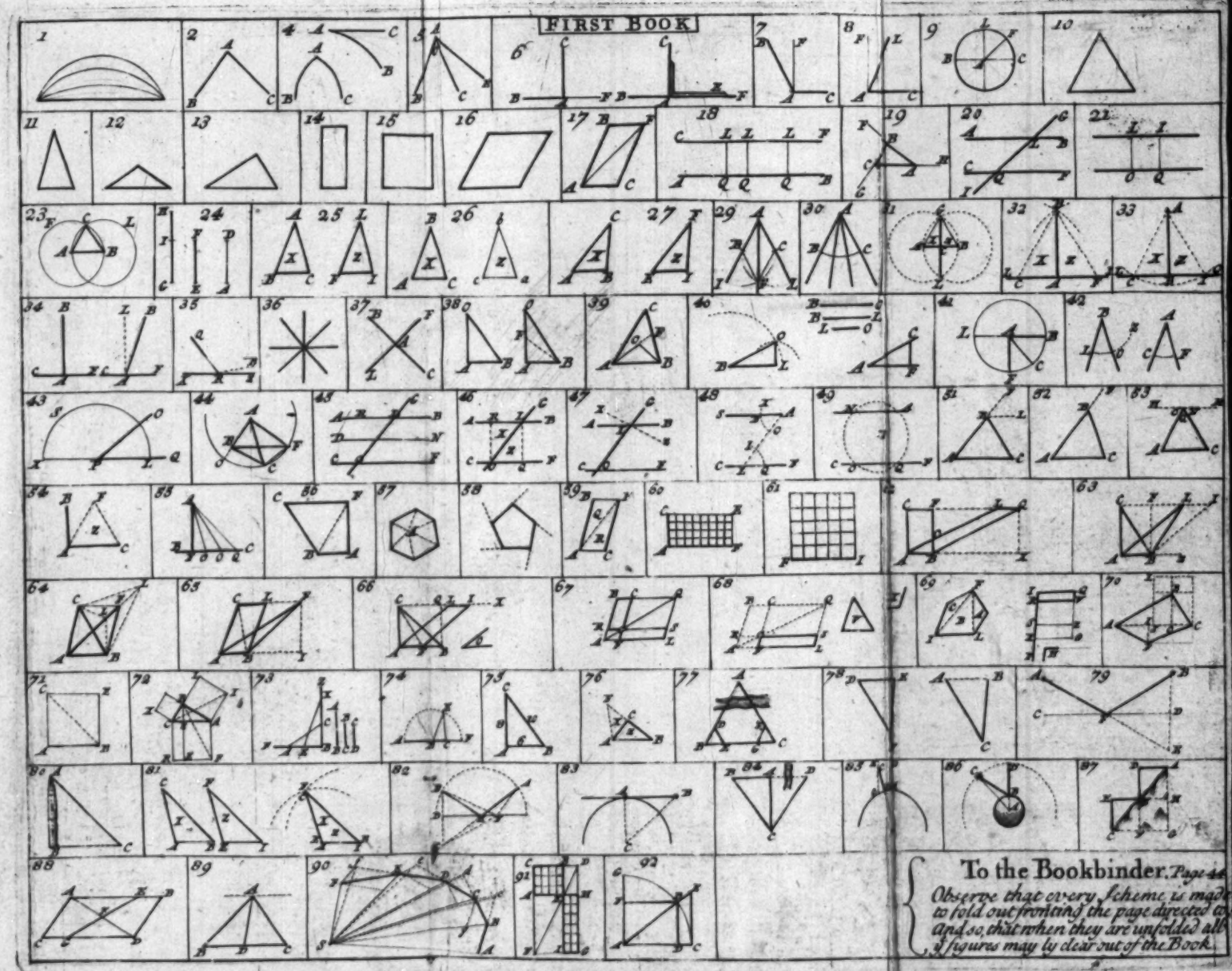
Fig. 76.

Sides (AC, BC) taken together, the Angle (ACB) which the two other Sides contain, is a right Angle.

If not, the Angle ACB will be greater or less than a right Angle. In either of which Cases (as it will be demonstrated, *Prop.* 12, 13, *l.* 2. which Propositions depend not on this) the Square of AB will not be equal to the Squares of AC, BC together; which is contrary to the Hypothesis.

Or thus. Draw FC perpendicular to AC, and equal to CB, and join AF. The Square of AF is [a] equal to the Squares of FC, CA together; that is, [b] to the Squares of BC, CA; that is by the Hypothesis, to the Square of AB. Therefore the right Lines AF, AB are equal. Because therefore the Triangles are mutually equilateral, the Angles at C [c] are equal. Therefore they are both right Angles [d]. Q. E. D.

[a] Per 47. l. 1.
[b] By the Construction
[c] Per 8. l. 1.
[d] Per def. 14.



To the Bookbinder. *Page 44*

Observe that every Scheme is made to fold out fronting the page directed to, and so, that when they are unfolded all the figures may lye clear out of the Book.

The Elements of *Euclid.*

BOOK II.

THIS Book is ſmall in Bulk, but great indeed in the Excellence and Uſefulneſs of its Thorems. Young Beginners will not, I know what I ſay, be at firſt able to diſcover it; but being further advanced, they will from their own Experience and with the greateſt Certainty apprehend that it is moſt true.

A DEFINITION.

A Right-angled Parallelogram (as A E) (which is wont ſimply, and without any Addition, to be call'd a Rectangle) is ſaid to be contain'd under the two Lines (A C, A F) which determine the Magnitude of it: *Fig. 60. l. 1.*

For the one of them A C determines the Heighth, the other A F the Breadth of it. Now if the Side A C be underſtood to be carried perpendicularly along the whole A F, or A F along A C, by that Motion the Rectangle or its Area will be produc'd. Wherefore a Rectangle is rightly ſaid to be produced from the Drawing of two Lines into one another, or the Multiplication of them one by the other. When therefore you have theſe Words, [the Rectangle under (or of) A C, C B,] or for Brevity's ſake, [the Rectangle A C B,] there is meant that Rectangle which is contained under A C and C B multiply'd one into the other. In like manner, when we ſay the Rectangle under A B, B C, or the Rectangle A B C, there is deſigned the Rectangle contain'd under the right Lines A B and B C, multiply'd by one another. *Fig. 2. l. 2.*

Moreover of Rectangles some are Oblong, some are Square. The Oblong Rectangle is that which hath its contiguous Sides unequal, or which is contained under two unequal right Lines. The square Rectangle that which is contain'd under two equal right Lines.

PROPOSITION I. Theorem.

Fig. 1. l. 2.

IF there be two right Lines (AB, AC) one whereof is divided into as many Parts as you will (AE, EF, FC) the Rectangle compriz'd under those two (AB, AC) is equal to all the Rectangles together, which are contain'd under the undivided Line (AB) and the several Parts of the divided Line (AE, EF, FC).

Make AB perpendicular to AC, through B draw the infinite Line BR parallel to AC. From E, F, C, erect the Perpendiculars EI, FL, CQ. BC will be a Rectangle under AB and AC; and is equal to the Rectangles BE, IF, LC; that is, (because as well IE as LF are equal (*a*) to AB) equal to the Rectangles under AB, AE; AB, EF; AB, FC.

(a) Per 29 & 34. l. 1.

Scholium.

THE ten first Theorems of this Book are true also in Numbers, if they as Lines be divided into Parts. The numerical Rectangles are produced from the Multiplication of two Numbers, and the numerical Squares from the Multiplication of the same Number by it self.

[*Let the undivided Number be 9, and the divided one 12. The Rectangle which is from 9 multiplying 12=108 will be equal to the three Rectangles, 27, 36, and 45, which are produced from 9 multiplied by 3, and 4, and 5, respectively and separately. Or let the Number 432 be as it were a multiplicand divided into 400 and 30 and 2; and the Number 8 an undivided Multiplier;* $8 \times 432 = 3456$ *will be equal to* $8 \times 400 = 3200 + 8 \times 30 = 240 + 8 \times 2 = 16$. *And from this Proposition therefore the Demonstration of Multiplication is to be deriv'd.*

PROP.

PROP. II. Theorem.

IF the right Line (AB) be cut any where (as in C), the two Rectangles under the whole (AB) and the Parts (AC, CB) are equal to the Square of the whole Line (AB). Fig. 2.

[*For AD is the Square of the whole; and AH, CD are Rectangles under the Whole AB and the Parts AC, CB.*] Fig. 17.

[*Let the Number 8 be divided into 5 and 3; the Square of the Whole* 8 × 8 = 64, *is equal to the Rectangles* 8 × 3 = 24, + 8×5=40.]

PROP. III. Theorem.

LET a right Line as (AB) be cut any where (as for Instance in C); the Rectangle contain'd under the whole AB and either of the Parts (BC) is equal to a Rectangle under the Parts (AC, CB) together with the Square of the said Part (BC). Fig. 3.

[*For AF is the Rectangle under the whole Line AB, and the Part AC; and CF is the Rectangle under the Parts; an AE is the Square of the Part AC.*] Fig. 28.

In Numbers. Let the Number 7 be divided into the Parts 3 and 4. The Rectangle of 7×3=21 *is equal to the Rectangle of* 3×4=12, *together with the Square* 3×3=9. *In like manner* 7×4=28, *is equal to the Rectangle* 3×4=12 + *the Square* 4×4=16.]

PROP. IV. Theorem.

LET a right Line as (FL) be cut any where, as in (O): The Square of the whole shall be equal to the Squares of the Parts (FO, OL) and to two Rectangles contain'd under the Parts (FO, OL). Fig. 4.

[*For FD is the Square of the whole; and CG and CL the Squares of the Parts; and CF, CD, two Rectangles under the Parts.*] Fig. 19.

In Numbers. Let the Number 10 *be divided into two Parts* 7 *and* 3. *The Square of* 10×10=100 *is equal to the Squares of the Parts* 7×7=49, *and* 3×3=9, *and to the two Rectangles* 7×3=21, *and* 7×3=21. *And on this Proposition depends the Extraction of the Square Root.*

Fig. 19. Coroll. (1.) *Hence it is manifest, that the Parallelograms about the Diameter of a Square,* (*O I, H K*) *are Squares.*

(2.) *As likewise that the Diameter of every Square bisects the Angles of it.*

(3.) *And that the Square of half of the Line is a fourth Part of the Square of the whole Line. For in that Case the Rectangles and Squares end in four equal Squares.*]

PROP. V. Theorem.

Fig. 5. *IF a right Line as* (*Q X*) *be cut equally in* (*R*), *and unequally in* (*S*) *the Rectangle contain'd under the unequal Parts* (*Q S, S X*) *taken together with the Square of the intermediate Part* (*R S*) *shall be equal to the Square of the half* (*Q R*).

Fig. 20. [*For Q H is the Rectangle under the unequal Parts, and L G the Square of the intermediate Part, and R F the Square of half the Line; and therefore, because the Rectangle Q L is equal to the Rectangle S F, and the rest of the Space is common to both, the Proposition is manifest.*]

Let the Number 8 *be divided equally, that is, into* 4 *and* 4, *and unequally into* 5 *and* 3. *The Rectangle of* 5×3=15 *together with the Square* 1×1=1 *shall be equal to the Square* 4×4=16.]

PROP. VI. Theorem.

Fig. 6. *IF a right Line* (*A B*) *be divided into two equal Parts in* (C), *and to it a certain right Line* (*B F*) *be adjoin'd; the Rectangle contain'd under the whole compound Line* (*A F*) *and the adjoin'd one* (*B F*) *taken together with the Square of half the Line* (C B) *shall be equal to the Square of*

(CF) which is compounded of half the Line AB and the adjoin'd one.

[*For AN is the Rectangle under the whole compound Line and the adjoin'd one; and KG the Square of half the Line AB; and CE the Square of the Line compounded of half the Line AB, and that which was added. Wherefore because the Rectangle HE is equal to the Rectangle AK, and the rest of the Space is common to both, AN & KG is equal to CE.* Q. E. D.] Fig. 21.

[*If the Number 6 be divided into the two equal Parts 3 and 3; and to it be added the Number 2. The Rectangle of* $8 \times 2 = 16$, *taken together with the Square* $3 \times 3 = 9$ *shall be equal to the Square* $5 \times 5 = 25$.]

Coroll. *Hence, with* Maurolycus, *with one single Observation we learn to measure the Diameter of the Earth. Let the Altitude of the Mountain AD be known, and AB the Line touching the Earth be known by measuring. Let the Line DE be cut into two equal Parts in the Center C, and to it be added the Line AD. Now because the Rectangle under AE, AD, together with the Square of DC, is by this Proposition equal to the Square of AC, that is, equal to the* * *Squares of the Lines AB, BC: From hence it follows that if you take away on both Sides the Square of CD or CB, the Rectangle which is under AE, AD is equal to the Square of AB. Therefore let the known Square of AB be divided by the known Altitude of the Mountain AD, and the Quotient will give the Line AE. From which substract the known Altitude of the Mountain AD, the remaining Line DE will be the Diameter of the Earth.* Q. E. I. Fig. 22. * Per 17.l.1.

PROP. VII. Theorem.

IF a right Line (AB) be cut any where (as in C), the Square of the whole Line (AB) taken together with the Square of either of the Segments (AC) is equal to two Rectangles contained under the whole (AB), and that Segment (AC) together with the Square of the other Segment (CB). Fig. 7.

[*For EB is the Square of the whole Line, and AL the Square of the Part AC. But the two Rectangles under* Fig. 23.

the whole Line and that Part, EI, HL, together with GB the Square of the other Part, possess the same Space that EB and the Square of AC doth. Therefore they are equal to EB & the Square of AC.]

[*Let the Number* 13 *be divided into any two Parts, as* 9 *and* 4. *The Square* 13×13=169, *together with that* 9×9 =81, *is equal to* 13×9=117, *and* 13×9=117, *and the Square* 4×4=16.]

PROP. VIII. Theorem.

Fig. 8.] *IF a right Line (LF) be divided into two equal Parts in (I) and to it a certain right Line be adjoin'd (FO); the Rectangle (LIO) which is contain'd under the half of the Line (LI) and the Line (IO) that is compounded of half the aforesaid Line, and the Line adjoin'd, this Rectangle taken four Times, together with the Square of the adjoin'd Line (FO), shall be equal to the Square of the whole compound Line (LO).*

Fig. 24. [*For AL is the Square of the whole Compound, containing four equal Rectangles under LI and IO (to wit, DR, BQ, RO, and the Fourth made up of LR and QH added together) and with those four Rectangles the Square HE. From whence the Proposition is manifest.*]

[*Let the Number* 12 *be divided into* 6 *and* 6; *and the Number* 4 *be added to it. The four Rectangles* 10×6=240 *and* 4×4=16 *are equal to the Square* 16×16=256.]

PROP. IX. Theorem.

Fig. 9. *IF a right Line (AC) be divided equally in (B) and unequally in (F), the Squares of the unequal Parts (AF, FC) will be double to the Squares of half the Line (AB), and of the intermediate Part (BF).*

Fig. 25. [*Le BE be equal and perpendicular to BA. From hence the Construction being made, as the Figure shews, the Lines AB,*

*AB, BE, BC will be equal: As also the Lines EG, GQ will be equal. The Angles AEC, ABE, CBE, EGQ, QFC, will be equal; and the Angles AEB, BEC, ECA, CQF, EQG half right ones. From whence the Square of AE will be double to * the Square of AB, which is half of AC, and the Square of EQ double to the Square of GQ or BF the intermediate Line. But the Squares of AE and EQ are † equal to the Square of AQ, that is, to the Squares of AF and FQ or FC the unequal Parts.* Q.E.D.]

* Per 47. l. 1.

† By the same.

[*Let the Number* 32 *be divided equally into* 16 *and* 16, *and unequally into* 20 *and* 12. *The Square* 20×20=400 *with the Square* 12×12=144, *are double to the Squares of* 16×16=256 *and* 4×4=16.]

PROP. X. Theorem.

IF a right Line (FI) be divided into two equal Parts in (L), and to it a certain right Line (as IO) be adjoin'd; the Square of the whole compound Line, (FO) taken together with the Square of the additional Line (IO), shall be double to the Squares, which are described upon the half Line (FL), and (LO) that which is compounded of half the Line (FI) and the additional Line.

Fig. 10.

[*For a Construction being supposed not unlike to the former; the Square of FE, is double to the Square * of the half Line FL, and the Square of EG is double to the * Square of EQ or LO which is compounded of the half Line and the additional one. But the Squares of FE and EG are * equal to the Square of FG; that is, to the Square of FO the whole compound Line; taken together with the Square of OG or OI the additional Line.* Q. E. D.

Fig. 26.

* Per 47. l. 1.

[*Let the Number* 40 *be divided into* 20 *and* 20, *and to it let there be added the Number* 14. *The Square* 54×54=2916, *with the Square* 14×14=196 *are double to the Square of* 20×20=400 *taken together with* 34×34=1156.]

PROP. XI. Problem.

Fig. 11.

SO to cut the given right Line (A B) in (C) that the Rectangle (A B C) which is contain'd under the whole Line and one Part, shall be equal to the Square of the other Part (A C).

From A erect a perpendicular A F equal to A B. Bisect A F in X. Draw the right Line X B; from the Line F A drawn forth cut off X I equal to X B. Then cut off A C equal to A I. I say the Thing is done.

For let the Square B A F S be perfected; and a Perpendicular being drawn through C, let the Rectangle F I L O be perfected also. Because F A is bisected in X, and to it is added A I; there shall be

(a) Per 6. l. 2.
(b) By the Construction
[c] Per 47. l. 1.

the Rect. F I A
\+ = (a) to the Square of X I
Square of X A

That is, = to the Square of X B (b)

That is, = to the Squares of B A, A X (d)

Therefore let there be taken away on both Sides the Square of X A; there will remain

Rectangle F I A or F L.

= A S the Square of the Line B A;

Wherefore again, the common Rectangle A O being taken away,

A L will remain equal to C S.

But A L is the Square of the Line A C, seeing by the Construction A C and A I are equal. And C S is the Rectangle A B C, forasmuch as B S is equal to A B. Therefore the Rectangle A B C is equal to the Square of A C. Therefore we have cut the Line A B, as it was required.

Scholium.

THE Ten first Propositions of this Book are true also in Numbers. But this Eleventh cannot be exemplify'd in Numbers; for no Number can be so divided that the Product of the whole multiplied by one Part shall be equal

equal to the Square of the other. The Force of this Section of a Line is wonderful, for which see *Prop.* 30. *lib.* 6.

PROP. XII. Theorem.

IN an obtuse-angled Triangle (ACB), the Square of the Side (AB) opposite to the obtuse-Angle (C), exceeds the Squares of the other Sides (AC, CB), by the Rectangle (BCF) twice taken; which same Rectangle is comprized under (BC), one of the Sides containing the obtuse Angle, and the Line (CF) which is intercepted betwixt the Perpendicular (AF) and the obtuse Angle. Fig. 12.

The Square of AB is equal to the Squares of AF, BF (*a*). (a) Per 47. l. 1.

But the Square of BF is equal to the Squares of FC, CB, with the Rectangle FCB twice taken (*b*). Therefore if you substitute these for the Square of BF; then (b) Per. 4. l. 2.

the Square of AB is equal to AF Square, FC Square, CB Square, and Rectangle BCF twice.

But the Squares of AF, FC are (*c*) equal to the Square of AC. Wherefore this being substituted for them, (c) Per 47. l. 1.

AB Square is equal to AC Square, CB Square, + Rectang. BCF twice.

PROP. XIII. Theorem.

IN any Triangle whatsoever (as ACB) the Square of the Side (AB) opposite to an acute Angle (C) is exceeded by the Squares of the other Sides (AC, CB) by the Rectangle (BCF) twice taken; which same Rectangle is contain'd under Fig. 13, 14.

(BC)

(BC) one of the Sides comprehending the acute Angle (C); and the Line (FC) which is intercepted betwixt the perpendicular (AF) let fall upon the Side (BC) from its opposite Angle (A), and the acute Angle (C).

[d] Per 4. l. 2. The Square of BC is equal to (*d*) the Rectan. BFC (twice, + FC Square + FB Square

[e] Per 47. l. 1. And, AC Square is equal to (*e*) CF Square + FA Square

Wherefore the two together { BC Squ. AC Squ. } are equal to Rect. BFC (twice + BF Square + 2FC Square + AF Square.

But the Rectangle BFC twice, together with the Square of FC twice, is (*a*) equal to the Rectangle BCF twice. Therefore this being substituted for them,

(a) Per 3. l. 2.

BC Squ. + AC Squ. are equal to the Rectang. BCF twice + BF Squar. + AF Squar.

But the Squares of AF, BF are equal to (*b*) the Square of AB. Therefore this being substituted for them,

(b) Per 47. l. 1.

BC Squ. + AC Squ. are equal to the Rectangle BCF twice + AB Squ.

That is, BC Square + AC Square do exceed AB Square by the Rectangle BCF twice taken.

Corollary.

Fig. 15. THE Proposition is true, although the Perpendicular falleth without the Triangle. And the Demonstration is almost the same.

[*More briefly thus.* $ACq = (c)\ ABq + CBq + 2CBF$. *Add on both Sides* CBq, *then* $ACq + CBq = ABq + 2CBq + 2CBF = (d)\ ABq + 2BCF$. Q. E. D.]

(c) Per 12. l. 2.
[d] Per 3. l. 2.

Scho-

Scholium.

FRom this Proposition and the 47th of the former Book, we have the Measure of any Triangle whatsoever, whose three Sides are known, although the Area be altogether inaccessible. For by the help of these Theorems, the Perpendicular is known, albeit the Impediments of the Place should not permit us to markit out. But Note, That the Perpendicular multiplied by half the Side on which it falls, produceth the Area of the Triangle, as appears out of the Scholium of Proposition 41. lib. 1.

Let there be any Triangle (as A B C) having its Sides known. It is required to give the Perpendicular A F, which falls from the given Angle A upon the opposite Side C B. *Fig.* 13, *or* 14.

Take the Square of the Side A B opposite to the acute Angle C, out of the Sum of the Squares of A C, and B C. By the 13th the Remainder shall be the Rectangle B C F twice taken. Divide half of the Remainder, that is, the Rectangle B C F by the known Side B C; thence will arise the right Line C F. Take the Square of the right Line C F out of the Square of A C. The Remainder will give (*a*) the Square of A F, whose Square Root will give the Perpendicular A F. [a] *Per prob.* 2. *scbol. post* 47. *lib.* 1.

This thing also may be obtained out of Proposition 12. But the 13th sufficeth, forasmuch as in every Triangle the Perpendicular let fall from some one of the Angles unto the opposite Side, falls within the Triangle.

PROP. XIV. Problem.

THE right-lin'd Figure (Q X Z) being given, to make a Square equal to it. *Fig.* 16.

Make (*b*) a Rectangular Parallelogram C I equal to the Rectilinear Q X Z; the Sides of which Parallelogram, if they shall be equal, you have already made the Square which was required; if they be unequal draw forth the greater Side I A unto L, until A L shall be equal to A C. Then bisect I L in Z; from which as from a Center [b] *Per* 45. *l.* 1.

ter through I and L describe a Circle, and let CA be produced till it meets the Circumference in B. The Square of the right Line A B is equal to the given Rectangle Q X Z.

For let the right Line Z B be drawn; because I L is cut equally in Z and unequally in A; the

Rectangle I A L
+ Z A Squ. } are equal (c) to Z L Squ, that is,
equal to (d) Z B Square, that is,
equal to (e) Z A Square
+ A B Square. }

(c) Per 5. l. 2.
[d] By the Construction
[e] Per 47. l. 1.

Taking away therefore on both Sides the common Z A *q*, there remains

Rect. I A L equal to A B *q*; that is,

Because A C and A L are equal, the Rect. C I equal to A B Square, and consequently A B Square equal to the rectilineal (*g*) Q X Z.

(g) By the Construction

Scholium.

EUCLID's Construction of this Problem requires that the given Rectilineal be reduced unto a Rectangle by *Prop.* 45. *l.* 1. Which Reduction being operose enough, the Problem perhaps will more readily be dispatch'd after this manner.

Let the given Rectilinear be resolv'd into as many Quadrangles (X, Z) as it can. Then to each Quadrangle (*a*) make an equal Rectangle. If there remain, as here it happens, one Triangle (Q), to it also (*b*) make a Rectangle equal. Then to each Rectangle by this 14. l. 2. make an equal Square; and lastly to all these Squares let one equal one be made (*c*). This will be equal to the given Rectilinear Q X Z.

[a] Per *schol.* *p.* 45. *l.* 1.
[b] Per Coroll. *p.* 42. *l.* 1.
[c] Per *prob.* 1. *schol.* *p.* 47. *l.* 1.

THE

The Elements of *Euclid*.

BOOK III.

THE fundamental Properties of the most perfect amongst plain Figures are demonstrated in this Book. The Usefulness of the Book is manifest by this one Thing alone, that it treats of a Circle, that abundant Source of admirable Things through the whole Mathematicks. The more famous Theorems are 16, 20, 21, 22, 31, 32, 35, 36.

DEFINITIONS.

1. THose Circles are equal whose Diameters or Semi-diameters are equal.

2. A right Line (F B) is said to touch a Circle, when it doth so meet it in the Point (B), that albeit it be produced it doth not cut it. *Fig.* 20. *l.*3.

3. Circles are said to touch one another, when they do so meet that they do not cut each other. *Fig.* 13. 14.

4. In a Circle the right Lines (B C, F L) are said to be equi-distant from the Center (A), when the Perpendiculars which are let fall upon them from the Center (A O, A I) are equal. *Fig.* 18.

5. Segments or Portions of a Circle are the Parts into which the right Line (C E) which cuts the Circle doth divide it. *Fig.* 37.

6. An Angle in a Segment is that (B Q C) which is contain'd under the right Lines, which are drawn unto one Point of the Circumference (Q) from the Ends of the Segment, (B, C). *Fig.* 33.

7. The Angle (C Q B) is said to stand upon the Circumference (B O C), as being opposite to it. *Fig.* 33.

8. A Se-

Fig. 11. 8. A Sector is that Part of a Circle which is contain'd by two Semidiameters, as (AB, AF), and an Arch as (BF or BCF) intercepted betwixt the Semidiameters.

PROPOSITION I. Problem.

Fig. 1. l. 3. TO *find the Center of a given Circle.*

Let the right Line (BC) be drawn in the Circle at random, which bisect in Q. Through Q draw the perpendicular LF, which bisect in A. A shall be the Center.

If you deny it; let the Center be O, which is without the right Line FL (for in FL it cannot be, forasmuch as this Line is divided every where unequally but in A): and let there be drawn BO, QO, CO. Because therefore you suppose O to be the Center, BO, CO must be equal; and the Triangles BOQ, COQ, must be equilateral to each other; seeing by the Construction BQ and CQ are equal, and QO is common. Therefore the Angle OQC (*a*) is equal to the Angle OQB. Therefore OQC is a right Angle (*b*), and consequently equal to LQC which is a right one by Construction, a Part to the whole. Which is absurd.

(a) *Per* 8. l. 1.
(b) *Per def.* 14. l. 1.

Corollary.

FRom what hath been demonstrated it appears, that if the right Line (LF) cuts another right Line BC into two equal Parts and perpendicularly, the Center is in the Line that cuts the other.

Fig. 2. The Center of a Circle is very easily found by a Square; the top of it (Q) being applied to the Circumference; for if the right Line DE joining the Points D and E in which the Sides of the Square cut the Circumference, be bisected in A, (A) shall be the Center. The Demonstration whereof depends on *Prop.* 31. *l.* 3.

PROP.

PROP. II. Theorem.

IF in the Circumference of a Circle there be taken two Points (B and C) the right Line which is drawn through them falls entirely within the Circle. Fig. 2.

Let there be taken in the Line BC any Point whatsoever, as O, and from the Center A, be drawn AB, AO, AC. Because AB, AC are equal, the Angles also B and C are (c) equal. Because therefore AOC is (d) greater than the internal one B, it shall be greater also than C. In the Triangle therefore OAC, the Side AC subtending the greater Angle AOC, is (e) greater than the Side AO subtending the lesser Angle C. Seeing therefore AC reaches no farther than from the Center to the Circumference, AO shall not reach so far. Therefore the Point O shall fall within the Circle. The same thing may be shew'd of any other Point of the Line BC. Therefore BC falls wholly within the Circle.

(c) *Per* 5. *l.* 1.
[d] *Per Corol.* 1. *prop.* 32. *l.* 1.
(d) *Per* 19. *l.* 1.

The Proposition is also manifest from the very Notion of a right Line and a Circle.

Coroll. *Hence it follows, that a right Line touching a Circle, toucheth it in one single Point only. For if it touched the Circumference in two Points, it would be a right Line drawn through two Points of the Circle, and consequently would fall within the Circle, contrary to the Definition of a Tangent. And by the like Reasoning (in passing from Planes to Solids) it might be prov'd, that every Plane toucheth a Sphere only in one Point.*

PROP. III. Theorem.

IF in a Circle a right Line (BL) drawn thro' the Center bisects another (CF) not drawn through the Center, it will cut it perpendicularly. And if it cut it perpendicularly, it will bisect it. Fig. 3.

Part 1. From the Center (A) let there be drawn AC, AF. The Triangles X and Z are equilateral to each other.

other. For CO, FO are by the Hypothesis equal, and AC, AF are so, because drawn from the Center; while AO is common to both. Therefore the Angles AOC, AOF are (*a*) equal. Therefore right (*b*) ones. Which was the first Part.

(a) Per 8. l. 1.
(b) Per def. 14. l. 1.
(c) Per 47. l. 1.

Part 2. Because by the Hypothesis AOC, AOF are equal Angles; AC Square shall (*c*) be equal to the Squares of AO, OC together; and AF Square equal to the Squares of AO, OF together. Seeing therefore the Squares of AC, AF are equal, the Squares of AO, OC together will be also equal to the Squares of AO, OF together: Wherefore taking away the common Square AO, the Squares of OC, OF remain equal. And therefore the right Lines OC, OF are equal. Which was the other Part.

Coroll. (1.) *Hence in every equilateral Triangle, and in that also which is only an* Isosceles, *a Line which falling from the Top of the Angle, bisects the Base, is perpendicular to it. And on the contrary, a Line which falling from the Top of the Angle is perpendicular to the Base, doth bisect it.*

(2.) *Hence it follows, that half of the Chord of every Arch, is the right Sine of half the Arch.*

PROP. IV. Theorem.

Fig. 4, 5.

IF in a Circle two right Lines (BC, FL), not drawn both of them through the Center, cut each other, they cannot bisect each the other.

Fig. 5.

For if one of them LF passeth through the Center, it is manifest that it shall not be bisected by BC which doth not pass through the Center.

Fig. 4.

If neither of them passes through the Center, from the Center A draw AO. If now BC, FL were both bisected in O, the Angles AOC, AOL would (*a*) be right Angles, and consequently equal; the whole to a Part, which is absurd.

(a) By the foregoing.

PROP. V, VI. Theorems.

Fig. 6, 7.

CIrcles cutting each other, or inwardly touching one the other, have not the same Center.

For

For if it were otherwise, the right Lines AB, ACF, drawn from the common Center A, would be equal; and AC would be equal to AF; a Part to the whole, because they are both equal to AB. Which is absurd.

PROP. VII. Theorem.

IF in a Circle there be taken any Point besides the Center (A), as the Point (C), and divers right Lines fall from thence unto the Circumference (as CB, CL, CO, CF); Fig. 8.

1. *(CB) which passeth through the Center will be the greatest.*

2. *The remaining Part of the Diameter (CF) will be the least.*

3. *Of the rest that will be the greater, which is nearer to the greatest.*

4. *And no more than two equal Lines can be drawn from the said Point (C) which is different from the Center, unto the Circumference.*

Part 1. Let AL be drawn from the Center A. Because AL, AB are equal, the common Line AC being added to each. AC and AL together are equal to CB. But AL + AC are greater than LC (*b*). Therefore CB is greater than LC. In the same manner BC will be shew'd to be greater than any other. (b) *Per* 20. *l.* 1.

Part 2. From the Center A draw AO. AO (that is AF) is less than AC, CO (*c*). Therefore taking away the common Line AC, CO remains greater than CF. In the same manner CF is prov'd to be less than CQ, or any other. (c) By the same.

Part 3. In the Triangles COA, CLA, the Sides LA, AC, are equal to OA, AC, each to each. But the Angle LAC is greater than the Angle OAC. Therefore (*d*) the Base LC is greater than the Base OC. (d) *Per* 24. *l.* 1.

Part 4. This is manifest from what goes before. For if there could be three drawn equal, CO, CI, CQ, there would be two on the same Side equal: Which is contrary to Part 3.

Coroll. *By the like reasoning* Theodosius *gathered, that of the Arches of great Circles drawn upon the Surface of a Sphere, from any Point diverse from the Pole of a certain Circle, unto that Circle, the greatest is that which passeth through the Pole of that Circle; the least, that which is drawn unto the opposite Point; and of the rest, that is the greater which is nearest to the greatest; as also that no more than two equal Arches can be drawn from that Point unto the Circle. And in the like manner may the Reader reason of himself on some other of the Propositions of this Book; it being very easy to pass from Plains to Solids in these Argumentations.*

PROP. VIII. Theorem.

Fig. 9, 10. *IF from a Point (A) taken without a Circle, there be drawn unto the Circle the right Lines (AB, AC, AF) or (AO, AQ, AR);*

1. *Of those which fall upon the concave Circumference, the greatest is (AB) which passes through the Center (Z.)*

2. *Of the rest, that is the greater, which is nearer to the greatest (AB).*

3. *Of those which fall without the Circle or upon the convex Periphery, the least is (AO) which being produced would pass through the Center (Z).*

4. *Of the rest, that which is nearer to the least is less than that which is farther off.*

5. *No more than two equal Lines can be drawn unto the Circumference from the same Point (A), whether they fall within the Circle, or only without.*

Fig. 9. Part 1. From Center Z draw ZC; because ZC, ZB are equal, the common AZ being added to each, AZ, + ZC are equal to AB. But AZ, + ZC are (a) greater than AC. Therefore AB is greater than AC. In like manner AB will be shewed to be greater than any other whatsoever.

(a) Per 20. l. 1.

Part 2. Draw ZF. Because in the Triangles AZC, AZF, the Sides AZ, ZC are equal to AZ, ZF is each to

to each; but the Angle AZC is greater than AZF, therefore the Base AC (*b*) will be greater than the Base AF. (b) *Per* 24. *l.* 1. *Fig.* 10.

Part 3. Draw ZQ. The two Lines AQ, QZ are greater than AZ (*c*). Taking away therefore the Equals ZQ, ZO, there remains AQ greater than AO. In the same manner AO is prov'd less than any other. (c) *Per* 20. *l.* 1.

Part 4. Draw ZR. The right Lines AQ, QZ are less than AR, RZ (*d*); therefore the Equals, ZQ, ZR being taken away, AR remains greater than AQ. (d) *Per* 21. *l.* 1

Part 5. This is manifest from the four foregoing.

PROP. IX. Theorem.

IF from some Point within a Circle (as A) more than two equal right Lines can be drawn unto the Circumference; that Point is the Center. *Fig.* 11.

This is manifest from Part 4. of Proposition 7.

PROP X. Theorem.

Circles cut each other in two Points only. *Fig.* 12.

For let them cut, if it may be, in more (B, C, F.) From A the Center of the Circle LQ, let there be drawn to the Points B, C, F, the Lines, AB, AC, AF: these will be equal. Because therefore from the Point A within the Circle OS, there are drawn three equal Lines AB, AC, AF, unto its Circumference, A must also be the Center (*a*) of the Circle OS. Therefore the Circles LQ, OS, which cut one another, have the same Center. Which contradicts Proposition 5. (a) By the foregoing.

PROP. XI. Theorem.

IF two Circles touch each other inwardly, a right Line drawn through their Centers (A and I) passes through the Point of Contact (B). *Fig.* 13.

If you deny it, let the Centers have, if it may be, that Situation that a right Line passing through them shall fall

without the Contact B, cutting the Circles (in O and L); Let the Center be A and C; and join A B, C B. Because therefore C B, C O are equal, the common A C being added to each of them, A C, + C B shall be equal to A O. But A C, C B are (*b*) greater than A B, that is, than A L (*c*). Therefore also A O is greater than A L, a Part than the whole. Which is absurd.

(b) *Per* 20. *l.* 1.
(c) By the Definition of a Circle.

PROP. XII. Theorem.

Fig. 14.

IF Circles touch one another on the out-side, a right Line which joins the Centers must pass through the Point of Contact.

If it be denied, let the Centers be so plac'd, as for Instance in A and B, that the Line passing through them shall not pass through the Contact S, but cut the Circles in O and Q. Let the Points A, S, and B, S be joined. Then A S, B S together will (*d*) be greater than A B. But A S is (*e*) equal to A O, and B S equal to B Q. Therefore A O and B Q together will be greater than A B, a part than the whole. Which cannot be.

(d) *Per* 20. *l.* 1.
(e) By the Definition of a Circle.

[Coroll. *A right Line drawn from the Center of one of the Circles through the Point of Contact, will pass through the Center of the other.*]

PROP. XIII. Theorem.

Fig. 15, & 16.

CIrcles touch both one another, and a right Line, in a Point only.

Fig. 15.

For let two Circles touch one another inwardly in a Part of the Circumference L C, if it may be. Then a right Line drawn through the Centers A and B will (*f*) pass through the Point of Contact, as in C. Let there be drawn also A L, B L. Because therefore B L, B C are equal (for they are drawn from the Center B unto the Circumference O L C) the common Line A B being added A B, B L shall be equal to A C. But A C is equal to A L, for they are both drawn from the Center A unto the Circumference L Q C. Therefore A B, B L are equal to A L, contrary to *Prop.* 20. *l.* 1.

(c) *Per* 11. *l.* 3.

Then

Then let the two Circles touch one another on the outside, in the Arch O L, if it may be. The right Line A P joining the Centers will pass through the Point of Contact (*a*), as in O for Instance. Let A L, P L, be drawn. The two Sides of the Triangle A L, P L will be equal to A O, P O or the whole A P; contrary to *Prop.* 20. *l.* 1. Fig. 16. [a] Per 12. *l.* 3.

Lastly, let the right Line B F and the Circle touch each other, if it may be, in some Part (C E): Let there be drawn unto the Center the right Lines C A, E A. The Lines C A, E A will then be equal: And therefore the Triangle C A E is an *Isosceles*. Wherefore the Angles C and E (*b*) are acute. Aud therefore a Perpendicular let fall unto B F from the Center A will fall betwixt E and C, (*c*) as for Instance in D. There will therefore both A C and A E be equal to the Perpendicular A D, which is absurd, and contrary both to *Coroll.* 14. *p.* 32. and to *Prop.* 47. *l.* 1. (b) Per Corol. 11. *prop.* 32. *l.* 1. [c] Per Corol. 3. *prop.* 32. *l.* 1.

Corollary.

Circles whose Centers are in the same right Line, and which cut it in the same Point B, do touch one another in that Point only. Fig. 17.

This Proposition is manifest from the very Notion of the Lines which are compar'd together. For neither can a right Line and the curve Circumference of a Circle, or the divers Curvatures of unequal Circumferences, or two Curvatures both convex, agree as to any Part of themselves. But they would agree, if they touched one another in some entire and proper Part.

PROP. XIV. Theorem.

In a Circle equal right Lines (B C, L F) are equally distant from the Center (A). And what Lines are equi-distant from the Center are equal. Fig. 18.

From the Center (A) let there be drawn (A C, A F.) Likewise (A O, A I) at right Angles to B C, F L. Thus B C, F L shall be bisected (*d*) in O and I. [d] Per 3. *l.* 3.

Seeing therefore the whole Lines BC, FL are supposed equal, the halves also OC, IF must be equal, and consequently the Squares of them are also equal. Seeing therefore the Squares of AC, AF are equal, and the Square of AC is equal to OCq and OAq, as also he Square of AF is equal to IFq, and IAq(*a*): It follows that the two Squares OCq, OAq are equal to the two Squares IFq, IAq. Wherefore taking away the Squares of OC, IF (which before were shew'd to be equal) the Square of OA remains equal to the Square of AI. Therefore the Perpendiculars OA, AI, are equal. Therefore (*b*) BC, FL are equi-distant from the Center. Which was the first Part. Then for the Converse of it;

(a) Per 47. l. 1.

(b) Per defin. 4. l. 3.

If the Distances OA, AI are supposed equal, then the Squares of the equal right Lines being taken away, by the same Ratiocination it will be shew'd that the remaining Squares OCq, IFq are equal, and consequently that the right Lines OC, IF are equal, which seeing they are * halves of the right Lines BC, FL, these also must be equal. Which was the second Part.

* Per 3. l. 3.

PROP. XV. Theorem.

Fig. 19.

OF right Lines inscribed in a Circle, the greatest is the Diameter; and of the rest, that is the greatest, which is the nearest to the Center.

Let there be any Line, as RS different from the Diameter FL. From the Center draw AR, AS. The two AR, AS, which are equal to the Diameter, are (*c*) greater than RS. Therefore, *&c.*

(c) Per 20. l. 1.

Then let BI be nearer to the Center than XZ. From the Center unto them draw the Perpendiculars AC, AQ. AQ shall be greater (*d*) than AC. Take therefore AO equal to AC, and through O draw RS perpendicular to AO, which (*e*) will be equal to BI; and let AR, AS, AX, AZ be join'd. Because therefore A is the Center, the Sides AR, AS shall be equal to AX, AZ. But the Angle RAS is greater than the Angle XAZ. Therefore the Base RS, that is, BI, is greater than the Base XZ (*f*). Q. E. D.

[d] Per def. 4. l. 3.

(e) By the foregoing.

(f) Per 24. l. 1.

PROP.

PROP. XVI. Theorem.

A *Right Line* (IF) *which being drawn thro' the Point* (B), *the Extremity of the Diameter* (CB) *is perpendicular thereto, falleth all of it without the Circle, and toucheth it in* (B). *Neither can there any right Line be drawn betwixt it ſelf and the Circle unto the Point of Contact* (B) *but it ſhall cut the Circle.* Fig. 20.

Part 1. Let there be taken in the Line IBF any Point L, unto which from the Center A draw the Line AL. Becauſe in Triangle ABL, the Angle ABL is a right one, by the Hypotheſis, ALB ſhall be acute (*g*). Therefore AL which is oppoſite to the greater Angle B will be greater than AB which is oppoſite to the leſſer Angle L (*h*). But AB reacheth only to the Circumference. Therefore AL ſhall reach beyond the Circumference; and conſequently fall without the Circle. Which was the firſt Part.

[g] Per Coroll. 5. p. 32. l. 1.

[h] Per 19. l. 1.

Part 2. Below BF, if it may be, let RB fall wholly without the Circle. Becauſe FBA is a right Angle by the Hypotheſis RBA will be acute, and therefore AB is not perpendicular to BR Therefore let there be drawn from the Center A to BR the Perpendicular AO, which (*a*) will fall towards R and cut the Circle in Q. Therefore AB which is oppoſite to the greater Angle, AOB, is greater than AO, which is oppoſite to the leſſer, to wit, the acute Angle OBA. But AB is equal to AQ: therefore AQ alſo is greater than AO, a Part than the whole.

(a) Per Corol. 3. prop. 32. l. 1.

Corollary.

1. HEnce it appears again, that the Contact of a right Line and a circular one, is only in one Point. Fig. 20.

2. If from Centers taken in the ſame right Line infinitely protracted, there be deſcribed through B infinite Circles, as well leſſer than the firſt BSC, as greater; Fig. 17.

they shall all touch the right Line I F in the same one Point B.

3. Circles therefore growing into an Amplitude greater than any given one, approach always, even unto Infinity, nearer and nearer to the Tangent, but are never join'd to it, otherwise than in one single Point of Contact; which thing although it be most evident, is yet truly admirable.

Fig. 17. 4. From these Things it is manifest, that every geometrical Line whatsoever is infinitely divisible. For let there be drawn from some Point of the Diameter unto the Tangent the right Line A Q. Infinite Circles having Centers in the right Line B A infinitely produced, touch the right Line I F by *Coroll.* 2. of this, and one another by *Coroll. p.* 13. in one and the same Point B, and consequently are no where join'd either amongst themselves, or with the right Line I F but in the Point B only. Therefore it is necessary that they divide the right Line A Q into infinite Parts, that is, into Parts exceeding any Number assignable.

Fig. 20. 5. The Angle of Contingence or Contact L B Q, (that, to wit, which is contained under the Tangent and the Circumference) cannot be divided by any right Line.

Fig. 17. 6. Nevertheless by Circumferences touching the Line I F in the same Point, it may be divided and diminished infinitely. And in this and the third Corollary lies hid the whole Mystery of Asymptotes, that is, of a right Line approaching unto an Hyperbola, together with it self infinitely produced, unto a Distance less than any given one, yet never concurring with it.

PROP. XVII. Problem.

Fig. 26. *From the given Point (B) to draw a right Line which shall touch a given Circle (O Q).*

From A the Center of the given Circle let there be drawn unto the Point B the right Line A B, cutting the Periphery in O. From the Center A describe thro' B another Circle B C, and from O draw O C perpendicu-

dicular to A B, which may meet the other Circle in C. Draw C A meeting the Circle O Q in I. The right Line drawn from B unto I will touch the Circle O Q.

For becauſe the Sides B A, I A, are equal to the Sides C A, A O and the Angle A contain'd betwixt the equal Sides is common to both. In the Triangles I A B O A C the Angles A O C, A I B are alſo (*a*) equal. Therefore A I B is a right Angle. For A O C is a right one by the Conſtruction. Therefore B I (*b*) toucheth the Circle in I.

(a) *Per* 4. *l.* 1.
(b) *Per* 16. *l.* 3.

Scholium.

BY the 31ſt following, from the given Point O a Line touching a given Circle (B Q) may be well drawn thus: *Fig.* 27.

Let the right Line joining the given Point O and the Center A be biſected in P. Then from Center P thro' A and O deſcribe a Circle, meeting the given one in B. The right Line O B will touch the Circle.

For A B being join'd, the Angle A B O in the Semicircle is a right one by *Prop.* 31. Therefore by *Prop.* 16. O B toucheth the Circle B Q.

PROP. XVIII. Theorem.

IF a right Line (C L) touch a Circle, a right Line (A B) drawn from the Center (A) unto the Point of Contact (B) is perpendicular to the Tangent. *Fig.* 28.

If it be denied, let ſome other right Line (as A F) be the Perpendicular from the Center A. This will cut the Circle in O. Becauſe therefore the Angle A F B is ſuppoſed to be a right one, A B F (*c*) muſt be acute. Therefore A B (that is, A O) is greater than A F (*d*); a Part than the whole.

(c) *Per Coral.* 5. *p.* 32. *l.* 1.
(d) *Per* 19. *l.* 1.

PROP.

PROP. XIX. Theorem.

Fig. 29. *IF a Line (BC) touch the Circle, and from the Point of Contact (A) there be rais'd (AI) perpendicular to the Tangent, the Center will be in that Perpendicular.*

If you deny it, let the Center be without AI in Z; and from it let there be drawn unto the Contact the Line ZA. The Angle ZAC will be a right one (*e*) and therefore equal to the Angle IAC, which by the Hypothesis is a right one; that is, the Part will be equal to the Whole.

(e) By the foregoing.

PROP. XX. Theorem.

Fig. 30, 31, 32. *THE Angle at the Center (BAC) is double to the Angle (BFC) which is at the Circumference, when the same Arch (BC) is the Base of the Angles,*

Fig. 30. Here are three Cases. In the first Case, the Sides BA, BF coincide. And then because AF, AC drawn from the Center are equal, there will be in the Triangle Z, the Angles F and C equal (*a*). But BAC is equal to the two Angles F and C (*b*). Therefore BAC is double of F.

(a) Per 5. l. 1.
[b] Per 32. l. 1.

Fig. 31. In the second Case BA, CA fall within BF, CF, and then FAX being drawn, XAB by the first Case is double of XFB; and XAC double of XFC. Therefore the whole BAC is double of the whole BFC.

Fig. 32. In the third Case, BF cuts AC, and the Angle BAC is without the Triangle BFC. Here let FAL be drawn. By the first Case the whole LAC is double of the whole LFC, and LAB taken away is double of LFB taken away. Therefore the remaining Angle BAC is also double of the remaining one BFC *Q.E.D.*

Fig. 52. [Corollary, *Hence we gather that the Sides of every Triangle are to each other as the Sines of the Angles opposite to those Sides respectively. Let EFG be any Triangle; about which*

which let a Circle be understood to be circumscrib'd (c), *and from the Center of the Circle, let there be let down the Perpendiculars AB, AC, AD, which will* † *bisect the Subtenses. Now as EF is to EG, so* $\frac{1}{2}$ *EF (that is, EB) to* $\frac{1}{2}$ *EG (that is, ED.) But EB is the Sine of the Angle* ‖ *BAE, that is, of half the Angle EAF, that is, of the whole Angle EGF* * *opposite to the Side EF; and ED is the Sine of the Angle EAD, that is, of half the Angle EAG, that is, of the whole Angle EFG, which is opposite to the Side EG. Therefore EF is to EG, as the Sine of the Angle EGF, is to the Sine of the Angle EFG.* Q.E.D. *And from this one Proposition a great Part of Trigonometry is deduced. Which Thing will be worth our Observation.*

(c) Per 5. l. 4.

† Per 3. l. 3.

‖ Per Corol. 2. p. 3. l. 3.

* Per 20. l. 3.

Coroll. (2.) *From the former Corollary we learn to measure the distance of the Moon. For Astronomical Observations giving us the Angle of the diurnal Parallax* * *BCA, we find out the Distance of the Moon by the following Proportion. As the Sine of the Angle ACB, is to the Sine of the Angle ABC; so is the Semidiameter of the Earth BA, unto the Moon's Distance, AC.* Q. E. I.

Fig. 86. l. 1.

* Corol. 16. p. 32. l. 1.

Coroll. (3.) *From the second Corollary we learn also to measure the Distance of the Sun. For there being given by Astronomical Observations the Angle of the menstrual Parallax, (namely, that which is made when the Moon appears precisely bisected,) or the Angle ZEO, and together with this Angle the Moon's Distance ZO. We find the Distance of the Sun by this Analogy. As the Sine of the Angle ZEO, is to the Sine of the Angle EOZ; which Sine is the Radius: So is ZO, the Moon's Distance, unto ZE the Distance of the Sun.* Q. E. I.]

Fig. 54.

PROP. XXI. Theorem.

THE Angles (BQC, BFC) which in a Circle stand upon the same Arch (BOC); or which are in the same Segment (BQSC), are all equal among themselves.

Fig. 33.

Let first the Segment BQSC be greater than a Semicircle. From the Center A draw AB, AC. By the foregoing the Angle BAC at the Center is double of each BQC, BFC. Therefore they all BQC, BFC are equal (a). Q. E. D.

(a) Per axio. 6.

Then

Then let the Segment B Q C be equal to or less than a Semicircle. In the Triangles B Q I, C F I, because the Angles vertically opposite at I are equal (*b*), the Sum of the rest, Q and R will be equal to the Sum of the (*c*) rest F and O. Wherefore if from these equal Sums there be taken away the Angles R and O, which by the first Part are equal, as standing upon the same Arch Q F, the Angles which remain Q, F must be equal. *Q. E. D.*

Fig. 34.
(b) *Per.* 15. *l.* 1.
(c) *Per Corol.* 10, *p.*32. *l.* 1.

Coroll. *Hence we gather in Opticks that any Line B C to the Eye placed where you will in the Circumference of the Circle, whereof the Line is a Chord, appears of the same Magnitude; to wit, because it appears every where under an equal Angle B Q C.*

[Schol. *If of two equal Angles standing upon the same Arch, one of them be at the Circumference, the other also will be at the Circumference.*

If it be denied, B Q C shall either be equal to the Angle B I C on this Side the Circumference Q F, or to the Angle B E C, which is beyond the said Circumference. But the Angle B I C is (d) greater, and the Angle B E C (d) is less than the Angle B Q C. Therefore, &c.]

Fig. 33. 34.
(d) *Per Corol.* 1. *p.*32. *l.* 1.

PROP. XXII. Theorem.

Fig. 35.

IN any Quadrilateral inscribed in a Circle (A B C F) the opposite Angles make two right ones.

Let B F, C A be drawn. The Angle A B C with the (*a*) two O and X make two right Angles. But O is equal to I (*b*), because it stands upon the same Arch B C: And again X is equal to Z (*b*) because it stands upon the same Arch A B. Therefore A B C taken together with the two Angles I and Z, that is, with the whole opposite Angle A F C, makes two right Angles. *Q. E. D.*

(a) *Per* 32. *l.* 1.
(b) *Per* 21. *l.* 3.

[Corollary, (1.) *Hence if one Side of a Quadrilateral described in a Circle be protracted, the external Angle will be equal to the opposite Angle of the Quadrilateral; for the internal added to either of them makes two right Angles.*

(2.) *Likewise a Circle cannot be described about a Rhombus; because its opposite Angles either fall short of, or exceed two right Angles.*

(3.) *Like-*

(3.) *Likewise if in any Quadrilateral ABCF the opposite Angles F and B are equal to two right ones, a Circle may be described about it. For (a) a Circle will pass through any three Angles C, F, A, and this so that the * fourth be equal to B; which cannot be unless it doth indeed pass through the Point B †. Therefore it doth pass through it.*

(a) Per 5. l. 4.
* Per 22. l. 3.
† Per schol. pr. 21. l. 3.

PROP. XXIII, XXIV. Theorems,

ARE not necessary; and they treat of similar Segments, which cannot rightly be defin'd without Proportions.

PROP. XXV. Problem.

TO perfect a given Arch (ABC). Fig. 36.

Let there be subtended at Random the two right Lines AB, CB; which bisect in I and L. From I and L raise perpendiculars meeting one another in O. This shall be the Center of that Circle whereof ABC is a Portion.

For (a) the Center is both in the Line I X, and in the Line L Z. Therefore it is in their common Point O.

[a] Per Corol. pr. 1. l. 3.

The Practice. From the Center B taken in the Arch describe a Circle: and with the same Interval from other Centers in the Arch describe two other Circles, each of which cuts the former twice. Two right Lines drawn through the Intersections, and crossing each other in O, will give the Center.

PROP. XXVI, XXVII. Theorems.

IN equal Circles equal right Lines (CE, FI) subtend equal Archs; and if the Archs are equal the Subtenses are also equal. Fig. 37

These Two Propositions are plainly Axioms, and need no Demonstration.

[Coroll.

Fig. 55. [Coroll. (1.) *If in a Circle ABCD the Arch AB be equal to the Arch DC; AD will be parallel to BC. For AC being drawn, the Angles ACB, CAD, as standing on equal Arches, will be equal. Wherefore AD * is parallel to BC.* Q. E. D.

* Per 27. l. 1.

Fig. 56. (2.) *The right Line EF which is drawn from the Point A, the middle Point of some Arch and toucheth the Circle, is parallel to the right Line BC, which subtends that Arch. For from the Center D draw unto the Point of Contact A the right Line DA, and join DB, DC. The Side DG is common, and DB is equal to DC, and the Angle BDA equal to the Angle CDA, the Arches BA, CA being supposed to be equal. Therefore the Angles DGB, DGC are equal *, and consequently are right Angles. But the internal Angles GAE, GAF are also right Angles †. Therefore BC, EF are parallel *.* Q. E. D.

* Per 4. l. 1.

† Per 18. l. 3.

* Per 28. l. 1.

PROP. XXVIII, XXIX. Theorems.

Fig. 38. *IF in equal Circles, the Angles whether at the Centers (BAC, FLI) or at the Circumference (BOC, FSI) be equal; the Arches also (BXC, FZI) on which they stand are equal; and if the Arches be equal, the Angles also are equal.*

These two Propositions also are plainly Axioms, and need no Demonstration.

PROP. XXX. Problem.

Fig. 39. *TO bisect a given Arch (ABC).*

Draw AC, which bisect in O. From O draw the Perpendicular OB, meeting the Arch in B. I say the thing is done.

For let AB, BC be join'd. the Sides AO, OB are by the Construction equal to CO, OB; and the Angles at O are equal, as being right ones. Therefore the Bases AB, CB are equal (*a*). Therefore the Archs also (*b*) AB, BC are equal.

(a) Per 4. l. 1.

[c] Per 26. l. 3.

The

The Practice. From the Centers A and C describe with an equal Interval Arches cutting each other in the Points F and I, the right Line drawn through these Points will bisect the Arch A B C.

PROP. XXXI. Theorem.

THE Angle (BCF) in a Semicircle is a right one; that in a Segment greater than a Semicircle, is less than a right one; that in a Segment less than a Semicirle is greater than a right one. Fig. 40.

Part 1. From the Center A draw A C. Because A B and A C are equal, the Angles O and B are equal (c). For the same Cause the Angles I and F are equal. Therefore the Angle B C F is equal to B and F together. Seeing (a) therefore the three together make two right Angles, B C F which is half of two right Angles, is one right Angle. (c) *Per* 5. *l.* 1. (a) *Per* 32. *l.* 1.

Part 2. Let the Segment L O B F be greater than a Semicircle, and in it let there be the Angle F O L, and let L B the Diameter of the Circle be drawn. The Angle F O L is less than that B O L, which by Part 1. is a right one. Therefore, *&c.* Fig. 41.

Part 3. Let the Segment L O X be less than the Semicircle L O B, and X O L be the Angle in it. This will be greater than B O L which is a right one. Therefore, *&c.* Fig. 41.

Corollary. *Hence we may take a Proof of the Instrument, called a Square, whether it be exactly Rectangular or not. For in what Circle soever the Top of the Square is laid upon C, or any Point of the Circumference whatsoever, if the Sides of it do pass through the Points of the Diameter B and F, the Angle is a right one; otherwise not.* Fig. 40.

(2.) [*If the Sides of a Square be held continually upon the Points B and F, in the mean while that the Angle is moved round, first on one Side, then on the other, the Top of the Angle C will describe a Circumference of a Circle, whose Diameter is the Line B F.*]

(3.) *Hence we learn to raise a Perpendicular at the End of a Line. Let B C be the Line, C the Point given, from whence a Perpendicular is to be rais'd. From any Point whatsoever*

soever A as the Center, let a Circle be described passing thro' the Point C, and cutting BC in any Point, as B. If the Semidiameter BF be drawn, it is manifest that the Line CF is the Perpendicular required. Q. E. F.

Fig. 37. (4.) [*Hence it is manifest, that Circles touching one another inwardly, do cut all Lines, as AD proportionably; or so, that AE the Subtense of the lesser, is to AD the Subtense of the greater Circle, as AC the Diameter of the lesser, is to AB the Diameter of the greater. For there being drawn the Subtenses EC, BD, the Triangles EAC, DAB are equiangled. For the Angle A is common, and AEC, ADB are right ones, as being Angles in a Semicircle; and therefore ACE, ABD (b) are equal. The Triangles therefore are similar, by the fourth Proposition of the Sixth Book, and AC : AB :: AE : AD.* Q. E. D.

[b] *Per Corol. 9. p. 32. l. 1.*

Fig. 40. (5.) *In a right-angled Triangle BCF, if the Hypotenuse BF be bisected in A, the right Line AC cuts the Triangle into two equicrural ones ACB, ACF, and so a Circle described from the Center A through B must past through C, the top of the right Angle.*]

PROP. XXXII. Theorem.

Fig. 42, 43. *IF a right Line (CF) touch a Circle, and another (AB) which is drawn from the Point of Contact (A) cut it, the Angle made by the Tangent and the cutting Line, is equal to the Angle which is made in the alternate or opposite Segment.*

That is, the Angle CAB will be equal to the Angle L, which is made in the Segment ALB; and the Angle FAB will be equal to the Angle O, which is made in the Segment AOB. For

Fig. 42. First, let the Line AB pass through the Center. Here by *Prop.* 18. CAB is a right Angle: And by *Prop.* 31. L is also a right one. Therefore CAB and L are equal.

Fig. 43. Then let the Line AB not pass through the Center. Let the Line AQ therefore be drawn through the Center, and BQ be join'd. Because the Angle in the Semicircle ABQ (*a*) is a right one, BQA taken together with BAQ will make one right Angle (*b*). But CAQ

(a) *Per* 31. *l.* 3.

[b] *Per* 32. *l.* 1.

is

is also by *Prop.* 18. of this Book a right Angle. Therefore BQA with BAQ are equal to CAQ. The common Angle therefore BAQ being taken away, there remains BQA, which is equal to L (*c*) equal to CAB: Therefore L and CAB are equal: Which is the first Part to be proved. (c) Per 21. l. 3.

Then FAB and CAB make two right Angles (*d*), and in the Quadrilateral BOAL, the opposites L and O make two right Angles (*e*). Therefore the two FAB, CAB are equal to the two O and L. Therefore there being taken away on one Side CAB, on the other L, which have already been shew'd to be equal, there remains FAB equal to O. Which was the other Part to be proved. (d) Per 13. l. 1. (e) Per 22. l. 3.

PROP. XXXIII. Problem.

Upon a given Line (BC) to make a Segment of a Circle, in which the Angle shall be equal to any Angle given. Fig. 44.

First let there be an acute Angle given ABF, from B draw BL perpendicular to AB; And at C, the Extremity of the Line BC, make BCI equal to CBL (by 23. *l.* 1.) whose Side shall cut BL in I. From the Center I describe a Circle thro' B: This Circle will also pass through C (forasmuch as because of the Equality of the Angles at B and C, the Sides likewise CI, BI are (by 6. *l.* 1.) equal) and the Segment BQC shall contain an Angle equal to the given one ABF.

For because AB is perpendicular to the Diameter BL, AB will touch the Circle which BC cuts (*a*). Therefore the Angle in the Segment BQC is equal (*b*) to the Angle ABF. (a) Per 18. l. 3. (b) By the foregoing.

But if the Angle given be obtuse as RBC, do as before, and COB will be the Segment required.

PROP. XXXIV. Problem.

From a given Circle to take away a Segment containing an Angle equal to a given one. Fig. 45.

[e] Per 23. l. 1. Unto the Diameter of the Circle F A draw the Perpendicular B A L. Then (*e*) let A C be drawn, which may make the Angle B A C equal to that which is given. This Line A C shall cut off the Segment A Q C, whose Angle is equal to the given one: As is manifest from *Prop.* 32.

PROP. XXXV. Theorem.

Fig. 46, 47, 48. *IF in a Circle two right Lines (C L, B F) cut one another, the Rectangle (C O L) under the Segments of one is equal to the Rectangle (B O F), under the Segments of the other. For,*

If they intersect each other in A the Center of the Circle, the thing is manifest.

Fig. 46. If one of them C L passeth through the Center A, and bisects the other B F which doth not pass through the Center; it (*f*) cuts it perpendicularly, and so the Square of F O is the same with the Rectangle F O B. Let A F be drawn. Because C L is bisected in A and otherwise divided in O,

[f] Per 3. l. 3.

It will be thus,

[a] Per 5. l. 2. Rect. C O L + A O q. } will be equal to A L q. (*a*).

that is, to A F q.

that is, equal to A O q. + F O q. } (*b*).

[b] Per 47. l. 1. Therefore the common Square A O being taken away, there will remain

Rect. C O L equal to F O q. that is, to the Rect. F O B.

Fig. 47. Then if one of the right Lines C L passes through the Center, and cuts the other B F unequally in O, let a right Line drawn from the Center A cut B F into two equal Parts in I. In this Case A I B will be a right Angle (*c*). Now because C L is bisected in A, and otherwise in O, it will be thus,

[c] Per 3. l. 3.

[d] Per 5. l. 2. Rect. C O L + A O q. } will be equal to A L q (*d*) that is, to A B q. that is, to A I q.

$$\left.\begin{array}{l} \quad AIq. \\ + BIq. \end{array}\right\} (e).$$

[e] Per 47. l. 1.

But AOq is equal to OIq. + AIq. (*f*). Therefore,

(f) By the same.

$$\left.\begin{array}{l} \text{Rect. } COL \\ + OIq. \\ + AIq. \end{array}\right\} \text{ equal to } \left.\begin{array}{l} AIq. \\ + BIq. \end{array}\right\}$$

Therefore the common Square AI being taken away, there remains,

$$\left.\begin{array}{l} \text{Rect. } COL \\ + OIq. \end{array}\right\} = BIq.$$

But BI Square is equal to the Rectangle FOB, together with OI Square: (*g*) because FB is bisected in I, and otherwise cut in O. Therefore,

[g] Per 5. l. 2.

$$\left.\begin{array}{l} \text{Rect. } COL \\ + OIq. \end{array}\right\} \text{are equal to } \left.\begin{array}{l} \text{Rect. } FOB \\ + OIq. \end{array}\right\}$$

Therefore the common OIq. being taken away, there remains,

Rect. COL. = Rect. FOB.

But lastly, If neither of the Lines CL, FB passes through the Center; Through their common Intersection let there be drawn the right Line XZ, which passes through the Center. By what hath been just now demonstrated, both the Rectangle COL, and that FOB are equal to the Rectangle ZOX. Therefore COL, FOB are equal betwixt themselves. Fig. 48.

[*Or the Proposition may be demonstrated more easily and universally thus: Join AC and BD. Here because of the Equality of the Angles CEA, BED as being vertically opposite (a); and of the Angles C and B as being upon the same Arch AD* †; *the Triangles CEA, BED are equiangled* (per Corol. 9. p. 32. l. 1.) *Therefore* * *CE : EA :: EB : ED. Therefore CE × ED is equal to EA × EB* (per 16. l. 6.) Q. E. D. Fig. 58.

[a] Per 15. l. 1.
† Per 21. l. 3.
* Per 4. l. 6.

PROP. XXXVI. Theorem.

IF from (B) a Point given without a Circle there be drawn unto the Circle two right Lines one (BF) touching it, the other (BC) cutting it; the Rectangle (CBO) which is comprehended under the whole cutting Line (CB) and the Part (BO) Fig. 49, 50, 51.

which lies betwixt the Point (B) and the Circle, is equal to the Square of the Tangent (BF.)

1. If the cutting Line BC passes through the Center A, join AF. This with the Line FB will make a right Angle (*a*). And therefore because CO is bisected in A, and to it is added OB; it will be thus,

[a] Per 18. l. 3.

[b] Per 6. l. 2.

$$\left.\begin{array}{l}\text{Rect. CBO}\\ +\text{AOq.}\end{array}\right\}\text{ will be equal to ABq. (}b\text{) that is}$$

$$\left.\begin{array}{l}\text{to AFq.}\\ +\text{FBq.}\end{array}\right\}(c).$$

[c] Per 47. l. 1.

Therefore the equal Squares AOq. AFq. being taken away on both Sides, there remains,

$$\text{Rect. CBO, } = \text{BFq.}$$

Fig. 50, 51.

2. But then if CB doth not pass through the Center, let there be drawn AB, AF, AO, and AL, and let AL bisect OC in L. The Angle ALO is therefore a right one (*d*). Likewise AFB is a right Angle (*e*). Now because CO is bisected in L, and to it is added OB, it will be thus,

[d] Per 3. l. 3.

[e] Per 18. l. 3.

[f] Per 6. l. 2.

$$\left.\begin{array}{l}\text{Rect. CBO}\\ +\text{LOq.}\end{array}\right\} = \text{LBq. (}f\text{)}$$

Let there be added on both Sides AL Square, and then

$$\left.\begin{array}{l}\text{Rect. CBO}\\ +\text{LOq.}\\ +\text{ALq.}\end{array}\right\}\text{ equal to }\left.\begin{array}{l}\text{LBq.}\\ +\text{ALq.}\end{array}\right\}$$

[g] Per 47. l. 1.

(h) By the same.

(i) By the same.

But the Squares of LO, AL are equal (*g*) to the Square of AO, or AF; and the Squares of LB, AL are equal to the Square of AB (*h*). Therefore,

$$\left.\begin{array}{l}\text{Rect. CBO}\\ +\text{AFq.}\end{array}\right\} = \text{ABq. that is, (}i\text{)}$$

$$\left.\begin{array}{l}\text{to BFq.}\\ +\text{AFq.}\end{array}\right\}$$

Therefore the common Square, that of AF being taken away, there remains

Rect. CBO equal to the Square of BF. *Q. E. D.*

Fig. 59.

[*Or more easily and universally thus. Draw AB and BC. Now because of the Equality of the Angles A, and DBC; * and for that the Angle D is common to both; the Triangles BDC, ADB are equiangled †. And therefore (by 4. lib. 6.) AD : DB :: BD : DC. Wherefore the Rectangle ‖ AD × DC is equal to the Rectangle DB × DB, or DBq. Q. E. D.*]

* Per 32. l. 3.

† Per Corol. 9. p. 32. l. 1.

‖ Per 16. l. 1.

Corol.

Corollaries.

1. IF from the same Point B without the Circle, as many cutting Lines BC as you will be drawn, all the Rectangles CBO are equal amongst themselves. For each of them is equal to the Square of the Tangent BE. Fig. 52.

2. The right Lines, which from the same Point touch the Circle are equal. For each of their Squares is equal to the same Rectangle.

[3. *It is also clear, that from the same Point B taken without the Circle, there can only two Lines BF, BQ be drawn, which shall touch the Circle. For if a third be said to touch it, it must be equal to BF, or BQ, and therefore the same with one of them.*

4. *In every right-angled Triangle BFA, that is not also an Isosceles, the Rectangle arising from the Sum of the Hypotenuse, and one Side drawn into the Difference betwixt them, is equal to the Square of the other Side. For the Sum of the Hypotenuse BA, + AF or AC, is = BC. And their Difference is BA—AF= BA—AO=BO. And the other Side of the Triangle is BF. But the Rectangle CBO is equal to the Square of BF. Therefore, &c.*] Fig. 49.

PROP. XXXVII. Theorem.

IF the Rectangle under CB and OB be equal to the Square of BF, this must touch the Circle in F. Fig. 25.]

From B let there be drawn the Tangent BQ, and the right Lines EQ, EF being drawn from the Center E, unto the Points Q and F, let BE be joined. Because by the Supposition the Square of BF is equal to the Rectangle CBO, as is also the Square of BQ, by 36. of this Book, the Squares of BQ, BF shall be equal betwixt themselves, and consequently the right Lines BQ, BF are equal. Therefore the Triangles FEB, BEQ are equilateral each to other. Therefore the Angles Q, F are equal (*a*). But Q is a right Angle (*per* 18. [a] Per 8. l. 1.

18. *l.* 3.) Therefore F also is a right Angle. Therefore B F toucheth the Circle (*b*).

[b] Per 16. *l.* 3.

[Corollaries 1. *Hence the Angle E B F is equal to the Angle E B Q* (per 8. l. 1.)

(2.) *If two equal right Lines B F, B Q fall from some Point B upon the convex Circumference, and B F one of them toucheth the Circle, the other B Q must touch it also. For seeing B F, B Q are equal, their Squares are also equal. But B Fq is equal to C B O (a). Therefore B Q q = C B O (b). Therefore B Q also toucheth the Circle (c).*

(a) By the foregoing.
[b] Per axi. 1.
[c] By this Proposition.

Scholium [1.] *Seeing all Plains passing through the Center of the Earth, which all stand perpendicular upon the Horizon, do produce great and equal Circles upon the Earth's Surface, we shall here bring in some elegant Consectaries from thence, out of our Author in his Astronomy; which from the Nature of Circles may very easily be understood.*

(1.) *If in any Part the Surface of the Earth were perfectly plain, Men could no more stand upright upon it, than upon the Side of an Hill, saving in the Point of Contact only.*

(2.) *The Head of a Traveller performs a longer Way or Course than his Feet; Likewise he that is on Horseback, and goes the same way as a Footman, measures a greater or longer Space than he that is on Foot. As likewise in a Ship, the uppermost Part of the Mast runs over more Way than the lower Parts of it.*

(3.) *If any one should travel over the whole Circumference of the Earth, the Way gone over by his Head, would exceed that which was gone over by his Feet, by the Difference of Circumferences; or by the Circumference of a Circle, whose Semidiameter is the Man's own Stature.*

(4.) *If a Vessel full of Water be elevated perpendicularly, the Water will continually be running over, and yet it will remain full; namely, because the Surface of the Water is continually compressed into the Surface of a greater Sphere. Yea, if a Vessel be elevated continually higher and higher, the Surface of the Water which is contain'd in it, will continually descend and come nearer unto a Plain; unto which yet it will never actually come.*

(5.) *If a Vessel full of Water be carried directly downwards although nothing run over, yet it will cease to be full; namely, because the Surface of the Water swells continually into a Part of a lesser Sphere. From whence it follows,*

(6.) That

(6.) *That one and the same Vessel contains more Water at the Foot of a Mountain than at the Top; as likewise more in a subterraneous Cellar, than in a Chamber. To which Things add,*

(7.) *That two Strings on which two iron Balls hang perpendicular,* [*and consequently the Walls of Buildings erected perpendicularly*] *are not Parallel one to another, but Parts of Radius's meeting together in the Center of the Earth.*

Scholium [2.] *I think it not amiss to insert in this Place this following Problem also, which was communicated to me by a Friend, as demonstrated by me somewhat more briefly.* Fig. 62.

Through the two Points (B) and (C) in a given Circle (BDM) to draw the Circumference of a Circle which shall bisect the Circumference of the other given Circle.

Through the Center A and one of the given Points B let there be drawn the infinite right Line BAME. Unto which from the Center let there be erected the Perpendicular AD, and let BD be drawn. Let the the Line DE, be made perpendicular to BD, cutting the infinite Line BAME in the Point E. Lastly, let a Circle be drawn (a) through the three Points, B, C, E. I say the Thing is done. For, [a] Per 5. l. 4.

Let a Chord of the second Circle be drawn through either of the Intersections of the Circles, as G, and through A the Center of the first Circle, to wit, GAf; Let also the Diameter of the first Circle GAF be drawn. Then in the first Circle (by Corol. 1. Prop. 8. l. 6. *and by* Prop. 17. l. 6.) $AB \times AE = AD^q$, *that is, (because of the Equality of the Semidiameters, AD, AG, AF)* $= AG \times AF$. *And in the second Circle there will be (b)* $AB \times AE = AG \times Af$. *Therefore* $AF = Af$, *and the Points F, f coincide, and the Arch FDG is equal to the Arch FMG. Q. E. F.]* [b] Per 35. l. 3.

The Elements of *Euclid.*

BOOK IV.

THIS Book, which is wholly Problematical, teacheth by what Artifice, Figures, those which are ordinate or regular especially, may be inscribed in, and circumscribed about, Circles. There is very great Use of it in Building Fortifications; and from it as a Fountain have been derived those most excellent Tables of Sines, Tangents, and Secants, to the very great Benefit of the Mathematicks.

[*This Book is most useful for* Trigonometry. *For by inscribing Polygons in a Circle, we learn to frame Tables of Chords, Tangents, and Secants: by the Help of which we learn to measure the Magnitudes of Figures and Bodies. Neither without it can we duly distinguish the Aspects, as they call them, of the Stars, as the Quartile, Sextile,* &c.; *they wholly depending upon the Inscription of Polygons in a Circle. Neither can we otherwise Collect the Area (which is a certain Quadrature of a Circle,) than from the Area's or Squares of innumerable Polygons inscrib'd in, and circumscrib'd about, a Circle. And in like manner we know the duplicate Proportion of Circles amongst themselves, from the duplicate Proportion of Polygons inscribed in, or circumscrib'd about, Circles. And as for military Architecture, it makes so much Use of Polygons inscrib'd in Circles, that more than all other Sciences it may seem to be wholly owing this Book.*]

DEFINITIONS.

1. A Rectilineal Figure is said to be inscrib'd in a Circle, or to have a Circle circumscrib'd about it, when the Tops of all the Angles thereof are in the Circumference of the Circle.

2. A

2. A rectilineal Figure is said to be circumscrib'd about a Circle, or to have a Circle inscrib'd in it, when each one of its Sides toucheth the Circle.

3. An ordinate or regular Figure is that which is equilateral and equiangular.

PROPOSITION I. Problem.

TO *inscribe a right Line* (*A*) *which is not greater than the Diameter into a Circle* (*B D*). Fig. 1. l. 4.

Take in the Circumference any Point B. From the Center B with the Interval of the given Line A describe an Arch, cutting the Circle in C. Draw the right Line B C. I say the thing is done.

PROP. II. Problem.

TO *inscribe in a Circle a Triangle having equal Angles with a given one* (*X*). Fig. 2.

Let the Line E F touch the Circle in D. Let E D G be made (*a*) equal to the Angle C, and F D H equal to B; and join G H. I say the Thing is done. For (*b*) E D G is equal to H. H consequently is equal to the Angle C (*c*). And F D H is equal to (*d*) to G; and consequently G to B. Therefore G D H (*e*) is equal to the Angle A. Therefore what was required is done.

(a) Per 23. l. 1.
(b) Per 32. l. 3.
(c) By the Construction
(d) Per 32. l. 3.
[e] Per Corol. 9. p. 32. l. 1.

PROP. III. Problem.

TO *circumscribe about a Circle a Triangle, having equal Angles with a given one* (*I L K*). Fig. 3.

Let the Line I K be drawn forth on both Sides, so as to make the external Angles O and N. Make at the Center A, the Angles G A B, E A F equal to O, N respectively, which is done by 23. l. 1. Then in the Points

Points G, F, B, let three right Lines touch the Circle, meeting together in C, E, D. The Triangle C E D is circumscrib'd about the Circle, and is equi-angled to the given one I L K. For,

In the Quadrilateral C G A B, the Angles G and B are (*f*) both of them right ones. Therefore the remaining ones G A B, and C taken together do (*g*) make two right ones, and consequently are equal to the two together, O, I. Therefore the two G A B and O, which are equal by the Construction, being taken away, there remains C equal to I. In the same manner E will be prov'd equal to the Angle K. Therefore D and L are (*h*) likewise equal. That therefore is done which was demanded.

(f) *Per* 18. *l.* 3.
[g] *Per Theo.* 1. *schol. pr.* 32. *l.* 1.
[h] *Per Corol.* 9. *prop.* 32. *l.* 1.

For that the Tangents do concur is thus shew'd. The Angles O, I, and K, N are (*a*) equal to four right ones; and I, K are less than two right ones (*b*). Therefore O, N, (that is by the Construction G A B, and B A F) are greater than two right ones. Therefore G A F (*c*) is less than two right ones. Therefore G F falls between A and D. Therefore seeing A G D, and A F D are right Angles, D G F, and D F G are less than two right ones. Therefore C G D and E F D (*d*) meet together towards D. In the same manner it may be demonstrated that the rest concur.

(a) *Per* 13. *l.* 1.
[b] *Per* 32. *l.* 1.
(c) *Per Corol.* 3. *p.* 13. *l.* 1.
(d) *Per schol. pr.* 31. *l.* 1.

PROP. IV. Problem.

Fig. 3.

TO *incribe a Circle in a Triangle.*

Bisect the two Angles C and E with the Lines C A, E A, meeting together in A. From A draw the Perpendiculars, A B, A G, A F. A Circle described from the Center A through B, will pass also through G and F, and touch the three Sides of the Triangle.

For in the Triangles C A G, C A B, because the Angles A G C, A B C, and likewise those G C A, and B C A are equal by the Construction, and the Side A C is common, the Sides A G, A B (*e*) must be likewise equal. In like manner A B, A F may be shewn to be equal. Therefore the Circle described from the Center A, passeth through B, G, F. And because the Angles at those three Points

(e) *Per* 26. *l.* 1.

Points are equal, it toucheth (*f*) all the Sides of the Triangle. That therefore is done which was required. (f) *Per* 16. *l.* 3.

[*Hence the Sides of a Triangle being known, the Segments of them which are made from the Contacts of an inscribed Circle will be known. Let* DC *be* 12. DE 18. CE 16. DC *and* CE *will be* 28. *from which substract* 18=DE=DG+BE, *there remains* 10=CG+CB. *Therefore* CG *or* CB=5. *Consequently* EB *or* EF=11. *Wherefore* FD *or* DG=7.]

PROP. V. Problem.

TO *describe a Circle about a Triangle, or through three given Points* B, C, D, *not lying in a right Line, to describe a Circle.* Fig. 4.

Connect the given Points with two right Lines BC, CD, which bisect with the Perpendiculars EA, OA, meeting together in A. This will be the Center of a Circle which passeth through B, C, D.

Let the right Lines AC, AD, AB be drawn. By the Construction the Sides DO, OA are equal to these CO, OA; and the Angles at O are right ones. Therefore AD is equal to AC (*a*). In the same manner AB may be prov'd equal to AC. Therefore AD, AB (*b*) are equal. Therefore a Circle described from the Center A through B, will pass also through C and D. Which was the Thing required. (a) *Per* 4. *l.* 1. [b] *Per axio.* 1.

As for the Practice, it is sufficient to describe from B, C, D three equal Circles, intersecting each other; and through the Intersections to draw right Lines, these meeting one another will give the Center sought.

PROP. VI, VII. Problems.

TO *inscribe a Square in, and Circumscribe one about a Circle.* Fig. 5.

Let the Diameters BD, CE be drawn, cutting each other perpendicularly. The right Lines which join the Terms of these, inscribe a Square in a Circle.

The

The Demonstration is manifest from 4. l. 1. and 31. l. 3. Then let four Tangents be drawn touching the Circle in B, C, D, E, meeting together in I, F, G, H. The Figure I F G H is a Square, circumscrib'd about a Circle.

The Demonstration is manifest from 18. l. 3. with Coroll. 2. Prop. 36. l. 3. and 28, and 34. l. 1.

Scholium.

Fig. 5.

A Square describ'd about a Circle is double to that inscrib'd. For because the Angle B C D in the Semicircle (*c*) is a right one, the Square of B D (that is F I Square) shall be (*d*) equal to B C q + C D q, and therefore double to the Square of C D, *i. e.* to C D E B.

(c) Per 31. l. 3.
(d) Per 47. l. 1.

PROP. VIII, IX. Problems.

Fig. 6.

TO *inscribe a Circle in, and circumscribe one about a Square,* (*as* B C F E).

Let there be drawn the Diameters of the Square, cutting each other in A. From the Center A describe a Circle through B; this will also pass through E, F, C.

Then from the Center A draw A D perpendicular to B C; a Circle describ'd from the Center A through D, will touch all the Sides of the Square.

Part I. Because by the Hypothesis the Lines C B, E B are equal; the Angles B C E, B E C will be equal (*c*). But C B E is a right Angle by the Hypothesis. B C E therefore and B E C are half right ones (*d*). In the same manner C B F will be shew'd to be an half right Angle, as likewise the rest of the Angles; and so they are equal amongst themselves. Therefore in the Triangle B A C, seeing there are two equal Angles C B A, B C A, the right Lines A B and A C (*e*) are equal. In the like manner the right Lines A B, A E, A F may be shew'd to be equal. Therefore a Circle described from the Center A through B, passes through E, F, C.

(c) Per. 5. l. 1.
(d) Per Corol. 11. prop. 32. l. 1.
(e) Per 6. l. 1.

Part 2. From A let there be also drawn the Perpendiculars A G, A H, A I. Because in the Triangles G B A, D B A,

DBA, the Angles at D and G, as likewise those at B are equal, and the Side AB is common, the Sides AD, AG must be equal (*a*). In the same manner AG, AH, AI may be shew'd to be equal. Therefore a Circle describ'd from the Center A, which passeth through D, will also pass through G, H, I, and touch all the Sides of the Square (*b*). Because the Angles at D, G, H, I are right ones. Therefore we have done what was required.

(a) *Per* 26. *l.* 1.

(b) *Per* 16. *l.* 3.

PROP. X. Problem.

TO *make an Isosceles Triangle BAC, in which the Angle at the Base (ABC, or ACB) shall be double to that which is at the Top (A).* Fig. 7.

Let any right Line, what you will, as AB, be taken, which so cut in D (*c*) that the Rectangle ABD shall be equal to AD Square. Then from the Center A thro' B describe a Circle; in which inscribe (*d*) BC equal to AD, and join AC. BAC shall be the Triangle sought.

(c) *Per* 11. *l.* 2.

(d) *Per* 1. *l.* 4.

For let the right Line DC be drawn, and through A, D, C describe (*e*) a Circle. Because the Rectangle ABD is equal to the Square AD, (that is, BC) it it manifest, that BC (*f*) toucheth that Circle DO which CD cuts. Therefore the Angle BCD (*g*) is equal to the Angle A in the opposite Segment, and so the common Angle DCA being added, BCA must be equal to A+DCA. But because the Sides AB, AC are equal, ABC (*h*) is equal to the Angle ACB. Therefore the Angle ABC is also equal to A+DCA. But the external Angle also BDC is equal to the two internal ones (*i*) A+DCA. Therefore ABC, and BDC are equal. Therefore the Line DC is (*k*) equal to BC, (that is, by the Construction to DA). Therefore the Angles A and DCA (*l*) are equal. Wherefore the Angle ABC, which hath been shew'd equal to those two, shall be double to one A. That is done therefore which was required.

(e) *Per* 5. *l.* 4.

(f) *Per* 37. *l.* 3.

(g) *Per* 32. *l.* 3.

(h) *Per* 5. *l.* 1.

(i) *Per* 32. *l.* 1.

(k) *Per* 6. *l.* 1.

(l) *Per* 5. *l.* 1.

Coro!-

Corollary.

EAch of the Angles at the Base B and C in the *Isosceles* now framed, is two fifths of two right ones, or four fifths of one right one, and the remaining one A is one fifth of two right ones, or two fifths of one right one. As is manifest out of this Proposition taken together with that 32. l. 1.

PROP. XI. Problem.

Fig. 8, 7. TO *inscribe a regular Pentagon in a Circle.*

(a) By the foregoing. Let there be described (*a*) the Triangle B A C, having the Angles at the Base double to that at the Top. Inscribe a Triangle C A D equiangled to this in a Circle (*b*). Bisect the Angles at the Base A C D, A D C, with the right Lines C E, D B, cutting the Circle in E and B. The Points A, B, C, D, E, join'd by right Lines, will give an ordinate Pentagon inscrib'd in a Circle.

(b) Per 2. l. 4.

For from the Construction it appears that the Angles I, N, Q, S, O are equal. Wherefore the Arches subtended to them A E, E D, C D, C B, B A are also (*c*) equal. Therefore the right Lines subtended to those Arches shall also (*d*) be equal. The Pentagon therefore is equilateral. But it is also (*e*) equiangular, because its Angles B A E, A E D, &c. stand on equal Arches B C D E, A B C D, &c. That therefore is done which was required.

(c) Per 28 l. 3.
(d) Per 27. l. 3.
[e] Per 29. l. 3.

Corollary.

Fig. 8. THE Angle of a regular Pentagon make six fifths of one right Angle, or three fifths of two. For the three Angles at A seeing they are equal, as standing upon equal Arches, B C, C D, D E, and the middlemost of them by the Corollary foregoing is two fifths of one right Angle; the three together, that is, the Angle of the Pentagon it self must make six fifths of one right one.

[Scho-

[Scholium. *This holds universally, that Figures of an odd Number of Sides are inscribed in a Circle, by means of an* Isosceles *Triangle, whose equal Angles at the Base are multiple of those at the Top. But Figures of an even Number of Sides are inscribed by the means of* Isosceles *Triangles, whose Angles at the Base are each of them multiple sesquialteral of that which is at the Top.* Fig. 8.

As in the Isosceles *ACD, if the Angle C or D be threefold of A, the Side CD will be the Side of an Heptagon; If fourfold, it will be the Side of an Enneagon, &c. But if C or D shall be* $1\frac{1}{2}$ *of A, CD will be the Side of a Square; and if C shall be* $2\frac{1}{2}$ *of the Angle A, CD will subtend a sixth Part of the Circumference: In like manner, if C or D shall be* $3\frac{1}{2}$ *of the Angle A, CD shall be the Side of an Octagon, &c.*]

Scholium.

F*Uclid's* Inscription of a Pentagon is ingenious, but that of *Ptolomy*, which he delivers in the first Book of his *Almagest*, is much more expeditious: And it is this.

Let the Diameters ED, BF be drawn, cutting one perpendicularly in A. Bisect the Radius AD in C. From the Center C through B describe an Arch, meeting the Diameter ED in G. The right Line GB is the Side of a Pentagon, and AG of a Decagon. Fig. 12.

The Demonstration cannot be given here, for it depends upon the 13th Book of *Euclid*. See it in *Clavius*, in his Scholium, after *Prop.* 10. *l.* 13.

Problem.

UPON a given right Line (AB) to describe a regular Pentagon. Fig. 9.

Cut AB so in C (*a*) that the Rectangle, ABC may be equal to the Square of AD. From AB protracted on both Sides take away AD, BE, equal to the greater Segment AC. From the Centers A and D with the Interval AB describe two Arches, cutting each other in F. Likewise from the Centers B and E describe, with the same (a) *Per* 11. *l.* 2.

same Interval, two Arches cutting each other in G. And again, from the Centers G and F, with the same Interval, describe two others, cutting each other in I. The Points A, F, I, G, B, being join'd, will give a regular Polygon upon the given right Line A B.

That it is equilateral is manifest from the Construction; that it is equi-angled will be thus demonstrated. Let D F be drawn. It is manifest by the Construction, that ADF is an *Isosceles*. And the Base AD is the greater Segment of the Side D F, so divided, that the Rectangle of the whole and the lesser Side, is equal to the Square of the greater. (For D F is equal to A B, and A D equal to A C). Therefore the Angle D A F is two fifths of two right ones; by *Coroll. Prop.* 10. *l.* 4. Therefore the remaining Angle F A B is three fifths of two right ones, or six fifths of one right one (*b*); and therefore is an Angle of a regular Pentagon (*c*). In the same manner may it be shewn, that the Angle G B A is three fifths of two right ones, and so equal to F A B. From whence it is necessary, that the rest F, G, I, should be equal to these, as appears from their being equilateral to these, if the right Line F G be conceiv'd to be subtended.

(b) *Per* 13. *l.* 1.
[c] *Per Corol. Pr.* 11. *l.* 4.

PROP. XII. Problem.

Fig. 10.

TO *circumscribe an ordinate Pentagon about a Circle.*

Let there, by the foregoing, be inscrib'd the regular Pentagon G H I K M, and let there be drawn Tangents in the Points G, H, I, K, M, which may concur in B, C, D, E, F. I say the thing is done.

For from the Center A draw the right Lines, A G, A B, A H, A C, A I. Here because from the same Point B, B G, and B H touch the Circle, they (*a*) are equal. Therefore the Triangles G A B, B A H are equilateral to each other. Therefore (*b*) the Angles O, P, as likewise those Q, S, are equal. And therefore the whole Angle B is double to P, and the whole G A H double to S. For the same Reason the Angles C and H A I are double to T and N respectively. But G A H and H A I are equal (*c*), because they stand upon equal Arches, by Construction G H, H I. Therefore their halves S and N are

(a) *Per Corol.* 2. *p.* 36. *l.* 3.
(b) *Per* 8. *l.* 1.
[c] *Per* 29. 3.

N are also equal. Because therefore in the Triangles BAH, HAC, the two Angles S and N are equal, and those at H are both right Angles (*d*), and likewise the Side AH is common; therefore the Sides (*e*) BH, CH, as likewise the Angles P, T, are equal. In the same manner I might shew BG, FG to be equal. Therefore BF, CB which are double to the equals BG, BH, are also equal. In the same manner it may be shew'd that the rest of the Sides of the circumscribed Pentagon are equal. It is therefore equilateral; but it is also equiangled; for seeing it hath been shew'd that the Angles B and C are each of them double to the equals P, and T, they must also be equal betwixt themselves. And in the same manner of the rest. We have therefore described a regular Pentagon about a Circle. Which was the thing to be done.

(d) *Per* 18. *l.* 3.
[e] *Per* 26 *l.* 1.

In the same way any ordinate Figure whatsoever is describ'd about a Circle, that is, if a like Figure be first inscrib'd in the Circle.

PROP. XIII, XIV. Problems.

TO *inscribe a Circle in a regular Pentagon, and circumscribe one about it.* Fig. 11.

Bisect the two Angles of the Pentagon B, C, with the right Lines BN, CS, cutting each other in A. From A draw the Perpendicular AL.

A Circle describ'd from the Point A with the Interval AL touches all the Sides of the Pentagon; and a Circle describ'd from the same Point A with the Interval AB, passes also through the Points F, E, D, C.

Part I. In the Triangles DCA, BCA, because the Sides DC, CA, are equal to BC, CA, by the Hypothesis, and the Angles P and O are equal by the Construction, those also G and I will be equal by 4. l. 1. Now the whole also B and D are equal by the Hypothesis. Wherefore seeing the Angle G is half of B by the Construction; I also will be half of D. Therefore D is bisected by the right Line DM. For the same Cause the rest of the Angles of the Pentagon E, F, are bisected, and consequently all the half Angles are equal betwixt themselves. Now let the Perpendiculars be

drawn, AM, AS, AN, AR. For because in the Triangles LBA, MBA, the Angles G and BLA are equal to the Angles Q and BMA, by the Construction, and the Side BA is common, AL and AM must be also equal (*a*). In like manner I might shew that the rest of the Perpendiculars, AM, AN, AS, AR, are equal. A Circle therefore from the Center A passing through L, will likewise pass through M, S, N, R, and because the Angles at L, M, S, N, R, are right ones by the Construction, * it will touch the five Sides of the Pentagon. Which was the first Part.

[a] *Per* 26. *l.* 1.

* *Per* 16 *l.* 3.

Part 2. In the Triangle CAB because the Angles O and G have already been shewn to be equal, the Sides also AC, AB must be equal (*b*), and in the same manner, AB, AF, AE, AD, may be prov'd equal, and therefore a Circle from the Center A passing through B must pass also through C, D, E, F. Therefore we have both inscrib'd a Pentagon in a Circle, and circumscrib'd one about a Circle. *Q. E. F.*

(b) *Per* 6. *l.* 1.

[*In the same way, in any regular Figure whatsoever, a Circle may be inscrib'd, and circumscrib'd about it.*]

PROP. XV. Problem.

Fig. 13.

IN a given Circle to describe a regular Hexagon.

Let the Diameter FAB be drawn. From the Center B through A describe a Circle, cutting the given one in C and D. Likewise from the Center F thro' A a Circle cutting the given one in E and G. The six Points, B, C, E, F, G, D, connected by right Lines will give the Hexagon required.

From the Center let fall the right Lines AE, AC, AG, AD. It is manifest that the Triangles H, I, M, L are equilateral, both in themselves, and with one another (*c*). Then because the Angles CAB, EAF, each of them make one third of two right Angles (*per Corol.* 12. *p.* 32. *l.* 1.) and therefore do make both together two thirds of two right Angles; it remains (*d*) that EAC is one third of two right Angles; therefore the Angles EAC, CAB are equal. But the Sides also EA, AC, are equal to the Sides BA, AC. Therefore the Base EC

[c] *Per* 1. *l.* 1.

(d) *Per Corol.* 1. *p.* 13. *l.* 1.

(*per*

(*per* 4. *l.* 1.) is equal to the Base B C, that is, to the Radius A C by the Construction. Wherefore the Triangle N is also equilateral. And in the same manner the Triangle K may be shewn to be so. Because therefore all the six Triangles, H, I, K, L, M, N are equilateral; it is manifest that all the Sides, C B, B D, D G, G F, F E, E C, are equal one to another; and to the Radius A C. The Hexagon is therefore equilateral. But it is also equiangular, seeing each one of its Angles E, C, B, D, G, F, consists of two Angles of an equilateral Triangle. Therefore we have inscribed a regular Hexagon in the Circle.

Corollaries.

1. THE Side of an Hexagon inscrib'd in a Circle, [or a Chord of 60 Degrees] is equal to the Radius [and consequently the Sine of 30 Degrees is equal to half the Radius (*per Corol.* 2. *p.* 3. *l.* 3.)]

2. An Angle of a regular Hexagon is four thirds of one right Angle; as consisting of two Angles of an equilateral Triangle, each of which makes two thirds of a right Angle.

3. If there be drawn the Diameter P S, perpendicular to the other F B; and with the Interval of the Radius P A, from the Center P, and S, there be made Sections in O and Q, in R and T, the Points, P, E, O, F, R, G, S, D, T, B, Q, C connected with right Lines will give a Figure of 12 Sides, inscrib'd in a Circle with one Aperture of the Compasses. Which Thing is of great Service in Dialling. *Fig.* 14.

4. From what has been demonstrated we may easily describe an equilateral Triangle in a Circle. The Diameter F B being drawn, from the Center B through A describe the Arch C A D. The Points C, F, D, connected with right Lines, will give the Triangle sought. *Fig.* 14.

5. The Side C X D of the equilateral Triangle, cuts off from the Diameter B F perpendicular to it, a fourth Part thereof B X. For the Angles A C X, B C X, standing upon equal Arches G D, D B are equal (*per* 29. *l.* 3.) and the Sides A C, C X, are equal to the Sides

[a] Per 4. l. 1. BC, CX. Therefore AX, BX are equal (*a*). Therefore BX is the fourth Part of the Diameter BF.

Scholium 1. *Problem.*

Fig. 13. * Per †. l. 1. YOU may raise a regular Hexagon upon a right Line BC thus. Make an * equilateral Triangle CAB upon the given Line CB. From the Center A through B and C describe a Circle. This will contain an Hexagon upon the given right Line CB. The Thing is manifest from the Proposition, and *Coroll.* 1.

Theorem.

THE Square of a Side of an equilateral Triangle is treble to the Square of the Semidiameter of a Circle in which it is inscrib'd, and is to the Square of the whole Diameter, as 3 to 4.

Fig. 14. Let there be drawn the Semidiameter AD. The Square of FD is equal to FAq + DAq + the Rectangle FAX twice taken (*per* 12. *l.* 2.) But the Rectangle FAX twice taken is equal to the Square of the Semidiameter FA or DA: (For because AX, XB (*b*) are equal, the Rectangle FAX twice taken, is equal to the two Rectangles which are under FA, AX, and under FA and XB, that is, equal to the Rectangle FAB (*c*); that is, equal to FAq.) Therefore the FDq is treble to FAq or DAq the Square of the Semidiameter.

[b] Per Coroll. 5. foregoing.

[c] Per 1. l. 2.

Now because the Square of the whole Diameter is quadruple of the Square of FA the Semidiameter (*d*), it is manifest that the Square of FD is to the Square of the Diameter, as 3 to 4.

(d) Per Corol. 3. prop. 4. l. 2.

Hence it follows that a Side of an equilateral Triangle is to the Diameter, as the square Root of 3 is to 2, the square Root of 4; and therefore that those Lines are incommensurable.

PROP.

PROP. XVI. Problem.

TO *inscribe a regular Quindecagon in a Circle.* Fig. 15.

Inscribe in the Circle AC the Side of a Pentagon (*a*), and AD the Side of an equilateral Triangle, (*per Corol.* 4. *p.* 15. *l.* 4.) bisect the Arch CD in E. CE is the Side of the Quindecagon, or fifteen-angled Figure sought. (a) *Per* 11. *l.* 4.

For if the whole Circumference be suppos'd to be 15, the Arch AC will be 3. and the Arch AD 5, and therefore the Arch CD 2, and consequently CE 1.

Corollary.

BY this Method innumerable regular Figures may be inscrib'd in a Circle. For if AC, AD, the Sides of two regular Figures be inscrib'd in a Circle, the Difference of the Arches CD will contain so many Sides of a new regular Figure, as are the Unites whereby the Denominators of the former differ one from another. But the Denominator of the new Figure is had, if the Denominators of the former be multiplied one by the other. Fig. 15.

As if AD be the Side of a Square, and AC of a Decagon, the Difference of the Denominators is 6. Therefore the Arch CD contains 6 Sides of a new Figure. But the new Figure is of 40 Sides. For the Denominators 4 and 10 multiplied one by the other make 40.

Scholium.

THere hath not yet been found out the Art by which regular Figures of 7, 9, 11, 13, 17, *&c.* Sides may be inscribed in a Circle, by a Pair of Compasses and a Rule only. Forasmuch as that Inscription of Figures depends upon the Division of the Circumference into any given Parts, which thing is yet lacking. But if the Circumference of a Circle be divided into 360 Parts, you may in a mechanical way inscribe any regular Figures whatsoever in it, after this manner.

Problem 1.

Fig. 15. DIvide 360 Degrees (that is, the whole Circumference) by the Denominator of the Polygon to be inſcrib'd (*e. g.* a Nonangle.) Make at the Center the Angle A G K of ſo many Degrees as are the Units of the Quotient in the ſaid Diviſion. A K ſhall be the Side of the nine-angled Figure, which is required to be inſcrib'd in the Circle.

Problem 2.

Fig. 15. BUT upon a given right Line you may deſcribe any regular Figure whatſoever by the Help of the following Table.

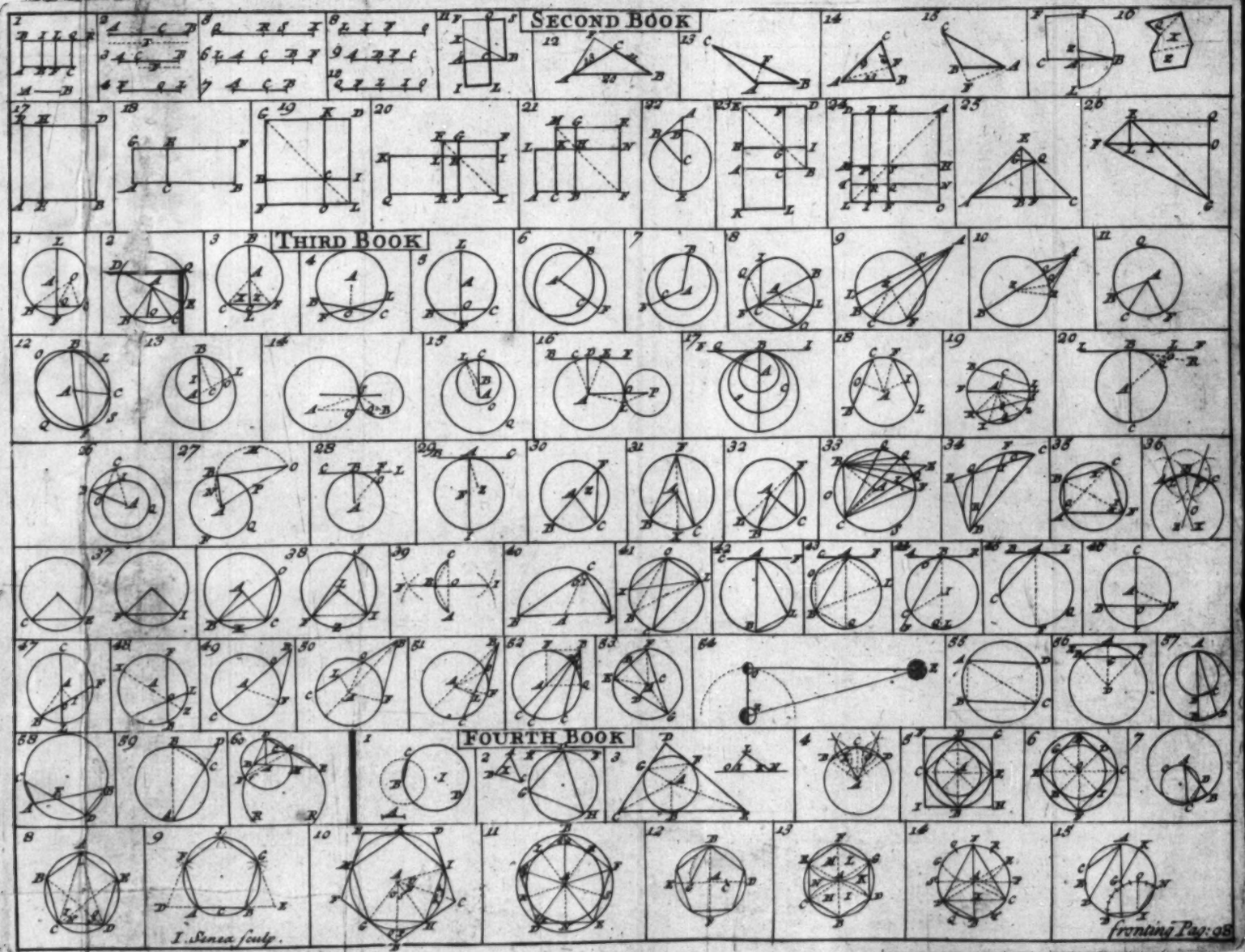
A right Angle is to the Angle of the Figure,

				Difference.
In a Pentagon as	5	to	6	— 1
In an Hexagon as	3	to	4	— 1
In an Heptagon as	7	to	10	— 3
In an Octagon as	2	to	3	— 1
In a Nonagon as	9	to	14	— 5
In a Decagon as	5	to	8	— 3
In an Undecagon as	11	to	18	— 7
In a Duedecagon as	3	to	5	— 2

Fig. 15. Let a regular Heptagon be to be inſcrib'd upon the given right Line X B. From the Center X with the Interval X B deſcribe a Circle, from which cut off the Quadrant B O. See in the Table what is the Proportion of a right Angle to the Angle of an Heptagon: You will find it to be as 7 to 10, and the Difference is 3. Divide the Quadrant therefore into ſeven equal Arches, ſo many of which add to it from O to N as the Difference hath Units. Through the three Points B, X, N, deſcribe (*per* 5. *l.* 4.) a Circle. This contains an Heptagon of the given right Line X B.

The Table was made by means of Theorem 2. in the *Schol.* upon *p.* 32. *l.* 1. by which is found the Number of right Angles, which the Angles of any right-lin'd Figure make; which Number being divided by the Denomina-

tor


SECOND BOOK
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FOURTH BOOK
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tor of the Figure, gives the Denominator of the Proportion of the Angle of the Figure to a right one.

Now becauſe hitherto many things have been propounded concerning regular Figures, let the following famous Theorem of *Proclus* cloſe this Book.

Theorem.

ONLY three regular Figures, to wit, 6 equilateral Triangles, 4 Squares, and 3 Hexagons, can fill a Space; that is, can conſtitute one continued Superficies. Which is thus demonſtrated. That ſome regular Figure often repeated ſhould be able to fill a Space; it is required that the Angles of many Figures of that kind being diſpoſed about one Point, ſhould make juſt four right ones; for juſt ſo many right Angles may be placed about one Point, as appears from *Corol.* 3. *Prop.* 13. *l.* 1. As for Example, that equilateral Triangles ſhould fill a Space, it is requir'd that ſo many Angles of ſuch Triangles N, M, L, K, I, H, being diſpos'd about the Point A, ſhould make juſt four right ones. But ſix Angles of an equilateral Triangle do make four right ones; (for one makes two thirds of one right one *, and therefore ſix of them make 12 thirds of one right one, that is, 4 right ones): Likewiſe the four Angles of a Square make four right ones, as is manifeſt; likewiſe three Angles of an Hexagon; for one maketh four thirds of one right Angle (*per Corol.* 2. *p.* 15. *l.* 4.); and therefore three of them do make twelve thirds of one right Angle, that is, four right ones. Therefore, *&c.*

Fig. 13.

* *Corol.* 12. *p.* 32. *l.* 1.

But that no other Figure beſides theſe can do this, will manifeſtly appear, if its Angle being found as above you ſhall multiply the ſame by any Number whatſoever; for the Angles will always either fall ſhort of, or exceed four right ones.

The Elements of *Euclid.*

BOOK V.

THIS fifth Book of Elements is altogether neceſſary for demonſtrating the Propoſitions of the Sixth Book. The Doctrine which it containeth is almoſt in continual Uſe. The way of Reaſoning from Geometrical Proportion is moſt ſubtle, ſolid and brief. This Method of Reaſoning, as a kind of Mathematical Logick, Geometry, Arithmetick, Muſick, Aſtronomy, Staticks, and all the other Parts of the Mathematicks, make eſpecial uſe of: Foraſmuch as they almoſt wholly depend upon Proportions connected together one with another; and are wont to borrow their Ways of Reaſoning concerning Proportionals from this Fifth Book. Practical Geometry, which conſiſts in the meaſuring of Lines, Figures, and Solids, is for the moſt part derived from the Doctrine of Proportions. There is not a Rule in Arithmetick but what may be demonſtrated from the Propoſitions of this fifth Book, without the Help of the 7th, 8th, and 9th Books, which treat profeſſedly of Numbers. We may fitly call the Muſick of the Ancients Geometrical Proportions apply'd to tuneful Sounds; which ſame Thing you may well nigh ſay concerning Staticks which are converſant about the Weights of Bodies. To comprehend the whole Matter in few Words, If you take away the Doctrine of Proportion from the Mathematicks, you will leave almoſt nothing which is excellent or greatly to be accounted of.

Scholium.

There is no Mathematician who is ignorant of how great Importance in Geometry the Knowledge of Proportions is; for it is the very Marrow, as it were, of the Mathematical Sciences; and the various Ways of Reaſoning concerning Proportionals

tionals, are both most useful and most certain; neither can we without them move one Step.

But then I reckon that this Doctrine is congenite in Men's Minds with common Reason it self; and that the various Ways of Reasoning concerning Proportianals, which Euclid *by much winding and going about delivers in this whole Book, do not so much need Demonstration, properly so call'd, as Illustration and Examples. And I altogether am of Opinion that those who take in Hand to deliver this most easy Doctrine by a long Circuit of Propositions, do involve a Thing in it self most clear, in a certain Cloud, and render it far more difficult. The Sum of the Matter I will open in a few Words. It is a thing easily known, that four Quantities are then proportional, or that the Analogies are then alike, when the first Quantity contains the second, as often as the third contains the fourth; or when the first is as often contain'd by the second, as the third is the fourth. So* 16 : 8 :: 4 : 2. *And* 3 : 9 :: 4 : 12. *Here are like or the same Proportions; because in the former Example the Consequents* 8 *and* 2 *are contain'd twice in their respective Antecedants; and so the Proportion of the Antecedents to the Consequents is double. And in the other Example the Proportions are also alike; because the Consequents* 9 *and* 12 *do contain their respective Antecedents three times; and so the Proportion of the Antecedents to the Consequents is subtriple. (Nor is there any Proportion of commensurable Quantities which may not be express'd by certain Numbers; nor indeed of Incommensurables, which may not be expressed by Numbers infinitely approaching nearer and nearer unto the true one.) Furthermore from what hath been said it appears, that like Proportions, whatsoever they are, may be express'd not only by divers Numbers, but also by the same. Thus* 2 *to* 1 *designs as well the Proportion of* 16 *to* 8, *as of* 4 *to* 2. 1 *to* 3 *no less expresseth that of* 4 *to* 12, *than that of* 3 *to* 9, *as is most manifest. Supposing therefore four Quantities to be proportional,* A : B :: a : b. *It is enquir'd in this Book after how many like Manners the Terms of these like Proportions may be changed, and ordered among st themselves. So that the emerging Proportion on both Sides may be still alike. And it may be answered, that it may be done after all the ways and manners possible; for seeing the Proportion of A to B, and that of a to b are alike, both of them may be express'd by the* same *Numbers after this manner,* A : B :: 9 : 3, *and* a : b :: 9 : 3. *And consequently all the Proportions emerging on both Sides, either by* Alternating *the Terms, or by* Inverting *them, or by* Com-

Compounding, *or* Dividing, *or* Converting, *or* mixing *them, may be expres'd by the* very same *Numbers*; *and consequently the same Proportion will always be kept on both Sides. As for Example sake.* $A+B : B :: a+b : b$, *because* $9+3 : 3$, *expresseth the same Proportion*; *which is* Composition. *The same is to be said of all the ways of changing the Terms. Therefore let Beginners observe this one Thing, that Proportions which are on both Sides the same be ever changed and ordered in the very same manner. And then there will be no Room to question, whether the Proportions which arise on both Sides be alike or no. It is indeed a Thing to be wonder'd at, that no one of those. who have hitherto compiled Elements of Geometry, have made use of this most easy Method of stating the Equality of Proportions, for the illustrating of this Fifth Book about the Doctrine of Proportions. Take therefore the primary Ways which Geometry make use of, in reasoning concerning like Proportions, as they are digested into this short Table.*

Let it be	A	:	B	::	a	:	b	::	9	:	3	
Then it will be by												
Alternating	A	:	a	::	B	:	b	::	9	:	9	:: 3 : 9
Inverting	B	:	A	::	b	:	a	::	3	:	9	
Compounding	$A+B$	:	B	::	$a+b$	:	b	::	$9+3$ (12)	:	3	
Dividing	$A-B$	:	B	::	$a-b$	:	b	::	$9-3$ (6)	:	3	
Converting	A	:	$A+B$	::	a	:	$a+b$	::	9	:	$9+3$ (12)	
or	A	:	$A-B$	::	a	:	$a-b$	::	9	:	$9-3$ (6)	
Mixing	$A+B$	:	$A-B$	::	$a+b$	:	$a-b$	::	$9+3$	:	$9-3$	

Ex æquo $A : B :: a : b$, *&* $B : C :: b : c$, *then* $A : C :: a : c$.
$9 : 3 :: 9 : 3$, $3 : 1 :: 3 : 1$, *then* $9 : 1 :: 9 : 1$.

Ex æquo perturbatè. $A : B :: a : b$, *&* $B : C :: r : a$, *then* $A : C :: r : b$.
$8 : 3 :: 8 : 3$, *&* $3 : 12 :: 2 : 8$, *then* $8 : 12 :: 2 : 3$.

Or thus, $8 : 3 :: 16 : 6$, *&* $3 : 2 :: 24 : 16$, *then* $8 : 2 :: 24 : 6$.
$a : b :: e+a : a+b$ *&* $b : e :: a+b : a+e$ *then* $a : e :: a+b : e+b$

He therefore that is expert in these Ways of Reasoning concerning Proportionals, and knows how to bring them into Use upon Occasion, will seldom stand in need of the particular Propositions of the Fifth Book. Only two of them, which yet are almost Axioms, may not improperly be inserted and illustrated by Examples, in way of Appendix, *because of the Frequency of their Use in all the Parts of the Mathematicks*; *which therefore shall be done after the Definitions.*

D E-

DEFINITIONS.

1. AN *Aliquot* Part of Magnitude, is that which being so many times more or less repeated, doth measure or is just equal to the Magnitude. An *Aliquant* Part is that which doth not measure it.

The Length of one Foot is an *Aliquot* Part of the Length of 10 Feet, because being ten times repeated it measures it. But the Length of four Feet is an *Aliquant* Part of a Line of 10 Feet, because being so many times repeated, to wit, twice, it falls short of it, but being thrice repeated it exceeds it.

2. One Magnitude is said to be a *multiple* of another, when the lesser measures the greater, and consequently is an Aliquot Part thereof; or when the greater contains the lesser so many times precisely.

3. Proportion is the mutual Respect, as to Quantity, of two Magnitudes of the same Kind.

Therefore there are in all Proportions two Terms, of which that is called the Antecedent which is first named, or which is nam'd in the Nominative Case; the other the Consequent.

When the Antecedent and the Consequent are equal it is called Proportion of Equality; when they are unequal, Proportion of Inequality.

4. Rational Proportion is that which is betwixt commensurable Magnitudes, and may be expressed in Numbers. Irrational Proportion, that which is betwixt Quantities incommensurable, and cannot be explicated by any Numbers.

Moreover, Commensurable Quantities are those which some common Measure measureth; Incommensurable, those which cannot be measured by any common Measure.

5. Two Proportions (that of A to B, and that of C to D) are alike, equal or the same; when the Antecedent of one (A) doth equally or in the same Manner (that is, neither more nor less, contain its Consequent (B), as the Antecedent of the other (C) contains its Consequent (F). *Fig.* 1. *l.* 5.

Or when the Antecedent of the one (A) is so often contain'd in its Consequent (B), as (C) the Antecedent of the other is in its Consequent (D). *Fig.* 2.

6. Two

Fig. 3. 6. Two Proportions are unlike, or one is greater than the other, when the Antecedent of one (I) doth more contain its Consequent (L), than the Antecedent of the
Fig. 4. other (O) doth contain its Consequent (Q); or when the Antecedent of one is less contain'd in its Consequent, than the Antecedent of the other in its Consequent.

7. Like or *similar* Parts are those which are equally or in the same Manner contain'd in their Wholes; so that what sort of Part one is of its Whole, such a Part the other is of its Whole. Which Thing indeed is nothing else, but that the Parts bear the same Proportion to their Wholes.

Aliquot Parts are like, which do equally measure their Wholes, as if each of them be one Third or one Tenth, *&c.* of its Whole.

Fig. 6. 8. Magnitudes (A, B, C, D) are said to be *continually* proportional when the middle Terms (B, C) are taken twice; that is, when they are each of them a Consequent in respect of the foregoing, and an Antecedent in respect of the following.

We thus pronounce continual Proportions. A is to B as B to C; and B is to C, as C is to D. And so on.

9. Magnitudes are *discretely* proportional when no Term is twice taken.

Fig. 1. Discrete Propositions we thus pronouce. A is to B as C to F. When there are more than three proportinal Magnitudes, if they be said to be proportional, they are always understood to be discretely so.

Fig. 6. 10. When the Magnitudes (A, B, C, D) are continually proportional, the first (A) is said to have to the third (C) a duplicate Proportion of that which it hath to the second (B): And the first (A) is said to have to the fourth (D) a triplicate Proportion of that, which the same first hath to the second (B): And so forwards.

[*If one triplicate Proportion be equal to another duplicate Proportion, the latter simple Proportion shall be sesquiplicate, or one and a half of the former simple Proportion. Let A, B, C, D, be* ∺; *and a, b, c,* ∺; *and let A the first in the former Analogy be unto D the fourth; as (a) the first in the second Analogy is to (c) the fourth; I say that (a) is to (b) in a Proportion which is one and a half of that which A is in to B. For let F be a middle Proportional betwixt B and C: Or, which is the same thing, betwixt A and D. Because of the*

the Equality of the Proportions of A to D, and (a) to (c), and the middle Proportions on both Sides F and (b); It will be A : F :: a : b. But the Proportion of A to F is compounded of the entire Proportion of A to B, and of the Proportion of the same B to C halved; and consequently the Proportion of (a) to (b), which is equal to that of A to F, contains the entire Proportion of A to B, and also the same halv'd, to wit, the Proportion of B to F. But the whole Proportion with its half, is a sesquiplicate or sesquialteral Proportion, or that which is one and a half of the other. (a) therefore is to (b) in a Proportion sesquiplicate of that of A to B. So in Astronomy, since the Cubes of the Distances of the Planets from the Sun bear that Proportion one to another, which the Squares of their periodical Times bear; so that the triplicated Proportion of the Distances, is the same with the Duplicate one of the periodical Times; It is wont to be said, that the periodical Times are in a sesquiplicate or sesquialteral Proportion of their Distances from the Sun.]

11. Antecedent Magnitudes are said to be Homologous or like to Antecedent, and Consequent to Consequent Magnitudes. As if A is to B, as C to F; A, C, and B, F, are homologous Quantities. *Fig. 1.*

XII. If a Set of Pairs of Quantities contain every one the same Proportion, that is the very Proportion also which the Sum of all the Antecedents bears to the Sum of all the Consequents.

$$\frac{20}{10}+\frac{6}{3}+\frac{8}{4}+\frac{18}{9}+\frac{4}{7}=\frac{66}{33}$$

XIX. If Parts be as Wholes, the Remainders will be also in the very same Proportion.

If 30 be to 20, as 3 to 2; 27 will be to 18 also, as 30 to 20, or as 3 to 2.

The

The Elements of *Euclid*.

BOOK VI.

THE Doctrine of Proportions, which was generally set forth in the Fifth Book, is applied in the Sixth, to plain Figures. And those Things which are delivered in this Book are so necessary to be known, that without them no Man can penetrate into the Secrets of Geometry, and reap the sweet Fruits of the Mathematicks. Each Proposition deserves to have an Encomium annexed; so great is the Utility of all.

This Sixth Book, as hath been said, begins to apply that excellent Doctrine concerning Geometrical Proportion, which was just before delivered, to divers and, those certainly, most notable Uses; and beginning with Triangles, the most simple of Figures, searches out their Sides and Areas, as they answer to one another in a certain Proportion. Then it defines proportional Lines, and the proportional Augmentations or Diminutions of Figures; and shews in what manner we may either increase or diminish them according to any Proportion given. It opens likewise the Golden Rule, or Rule of Proportion, the very chief of all Arithmetic; and demonstrates that in a rectangle Triangle, not only the Square, but also the Pentagon, Hexagon, and in general, every regular Polygon, which is described by the Hypotenuse, is equal to the Squares, Pentagons, Hexagons, or any regular Polygons whatsoever, that are describ'd by the two Sides. It also propounds most easy and certain Principles for measuring as well Solids, as Lines and Surfaces, which are of very great Use in all Parts of the Mathematicks.

DEFI-

DEFINITIONS.

1. LIke or similar Figures are those which both have all the Angles equal, each to each other respectively, and the Sides which are opposed to the equal Angles or which are betwixt them, or which are about the equal Angles, (for they come all to one) Proportional.

As the Triangles X, Z, will be said to be like, or similar, if the Angle A be equal to the Angle F, and the Angle B equal to the Angle I, and consequently the Angle C equal to the Angle L: And moreover if AB be to FI, as BC to LI; and BC is to LI, as CA is to LF; and CA to LF as AB to FI; by comparing always the Sides opposite to the equal Angles. In the same manner the Likeness of all right-lin'd Figures may be explained. *Fig. 7. l. 6.*

2. Reciprocal Figures are when the Antecedent and Consequent Terms of the Proportions appear on both Sides. *Fig. 29.*

As in the Parallelograms X, Z,
If AC be to CB,
As FC is to CL.

The Antecedents here are AC, and FC of which there is one in both Figures; and the Consequents are CB, and CL; of which likewise there is one in each Figure. And therefore the Parallelograms X, Z are call'd reciprocal. Understand the same of other Figures.

3. The Altitude of a Figure is the Perpendicular let fall from the Top to the Base. This is with *Euclid* the fourth Definition.

As the Altitude of the Triangle ABC is the perpendicular AQ which falls from the Top upon the Base BC, either within the Triangle or without, upon the Base protracted. Now the Base and Top are assumed at Pleasure. *Fig. 1.*

4. Like Arches of Circles are those which have the same Proportion unto their whole Circumferences.

As if each of them be a third or fourth Part, *&c.* of that Circumference.

PROP.

PROPOSITION I. Theorem.

Fig. 2. *TRIANGLES (ABC, DEF) and Parallelograms (AOPC, DQRF) which are betwixt the same Parallels, or have the same Altitude, have the same Proportion betwixt themselves as their Bases, (AC, DF.)*

Upon this Theorem the whole Sixth Book depends, yea, whatsoever any where is demonstrated about Figures by Proportions, whether Plain or Solid.

Let there be taken any Aliquot Part of the Base DF: *E.G.* DG one Third, and let the right Line GE be drawn: The Triangle DEG will likewise be one third Part of the Triangle DEF as is gathered from 38. l. 1. Wherefore DG and the Triangle DGE are like Aliquot Parts of their Consequents *. Then let there be taken away DG from the Base AC as often as it can, as suppose six times, and let the right Lines HB, IB, KB, LB, MB, NB, be drawn. Because the Lines CH, HI, *&c.* are each of them equal to DG, the six Triangles CBH, HBI, *&c.* are each of them (*a*) equal to the Triangle DEG. Therefore as often as DG is contain'd in the Antecedent AC, so often is the Triangle DEG contain'd in the Triangle ABC. By the same Reasoning it may be shew'd, that the like Aliquot Parts whatsoever of the Consequents (the Base DF, and the Triangle DEF) are in an equal Number contain'd in the Antecedents (the Base AC, and the Triangle ABC), therefore as the Base AC, is to the Base DF; so is the Triangle ABC, to the Triangle DEF. *Q.E.D.*

* *Per defin.* 7. *l.* 5.

(a) *Per* 38. *l.* 1.

But now because the Parallelograms AP, DR are (*b*) double to the Triangles ABC, DEF, they also will be as their Bases.

[b] *Per* 41. *l.* 1.

Corollary.

Fig. 3. THE Triangles (ABC, FIL) and the Parallelograms which have equal Bases (AC, FL) or the same, have that Proportion one to another, which their Altitudes (BO, IQ) have.

For

For let QS, OR, be made equal to the equal Bases (FL, AC); QS, OR will then be equal. Draw SI, RB. If in the Triangles OBR, QIS, you take BO, IQ for the Bases, OR, QS, will be their Altitudes; which seeng they are equal, the Triangles OBR, QIS (c) will be bewixt themselves, as their Bases BO, IQ. But because by the Construction OR is equal to AC, and QS equal to FL, the Triangles OBR, QIS, are (d) equal to the Triangles ABC, FIL. Therefore the Triangles ABC, FIL, are also as BO is to QI.

(c) Per. 1. l. 6.

(d) Per 38. l. 1.

Coroll. (2.) *Hence a Trapezium ABCD, whose Sides AD and BC are parallel, may be divided into any equal Parts whatsoever. For let CE be made equal to AD. Because of the Equality of the Angles vertically opposite* (e) *AFD, EFC, and of the alternate Angles* (f) *DAF, FEC, and ADF, ECF, and the Equality of the Bases AD, CE, by Construction, the Triangles ADF, FCE* (g), *are equal; and therefore the Triangle ABE is equal to the Trapezium ABCD. Therefore the Base BE being divided into any equal Parts whatsoever; as for Instance, three, BG, GR, RE, the Triangles ABG, AGR, ARE, shall each of them be one third Part of the Trapezium.* Q. E. I.

Fig. 50.

(e) Per 15. l. 1.

(f) Per 27. l. 1.

(g) Per 26. l. 1.

PROP. II. Theorem.

IF to one Side of a Triangle (as BC) there be drawn (FL) a Parallel, this cuts the Sides proportionally, that is, (AF) will be to (FB) as (AL) to (LC).

Fig. 4.

And if the right Line (FL) cuts the Sides (BA, CA) proportionally, it will be parallel to the other Side (BC).

Part 1. Let BL, CF be drawn. Beeause FL is supposed parallel to BC, the Triangles FBL, LCF having the same Base are (a) equal. Therefore the Triangle X hath the sameProportion to both; now the Triangle X, is to the Triangle FBL, as the same Triangle X is to that LCF. But the Triangle X is to the Triangle FBL (b), as AF is to FB; and the Triangle X is to that LCF as AL (c) is to LC. Therefore also AF is to FB, as AL to LC. Q. E. D.

(d) Per 37. l. 1.

(b) By the foregoing.

(c) By the same.

[d] By the foregoing.

Part 2. As A F is to F B, so is the (*d*) Triangle X to the Triangle F B L: And as A L is to L C, so is the same Triangle X to the Triangle L C F. Now A F is suppos'd to be to F B, as A L is to L C. Therefore the Triangle X is to the Triangle F B L, as the same X is to L C F. Therefore the Triangles F B L, L C F are equal. Therefore seeing they have a common Base F L, the Lines F L, B C, are (*e*) parallel. *Q. E. D.*

[e] Per 39. l. 1.

Corollary.

Fig. 5.

IF unto (B C) one Side of a Triangle there be drawn more Parallels (I O, F L) all the Segments of the Sides will be proportional.

Let F Q be drawn parallel to A C. The right Lines F S, S Q, are equal (*f*) to L O, O C. But B I is to F I as Q S to S F (*a*). Therefore B I is also to I F as C O to O L.

(f) Per 34. l. 1.
(a) Per 2. l. 6.

PROP. III. Theorem.

Fig. 6.

IF a right Line (B F) which bisects an Angle of a Triangle, doth also cut the Base (A C), the Segments of the Base (A F, F C) will have the same Proportion betwixt themselves as the Sides (A B, B C) have.

And if the Parts of the Base (A F, F C) have the same Proportion betwixt themselves, as the other Sides (A B, C B) the Line (B F) which cuts the Base, bisects the opposite Angle (A B C).

Part 1. Draw forth C B until B L be equal to B A; and join A L. Because in the Triangle Z the Sides L B, A B, are equal, the Angles also (*b*) L and O are equal. Because therefore the external Angle A B C is equal to the two internal ones (*c*) L, O, the Angle I which by the Hypothesis is half A B C, will be equal to the Angle L. Therefore A L, F B (*d*) are parallel. Therefore in the Triangle A C L, A F is to F C (*e*) as L B (that is, A B) is to B C. *Q. E. D.*

[b] Per 5. l. 1.
(c) Per 32. l. 1.
(d) Per 29. l. 1.
(e) Per 2. l. 6.

Part

Part 2. Protract CB again until BL be equal to BA. Because AF is suppos'd to be to FC, as AB (that is, LB) is to BC; AL, FB (*a*) are parallel. Therefore the external Angle I is equal to the internal one L (*b*); and the alternate Q equal to the alternate O. But because LB, AB, are equal, the Angles L and O (*c*) are equal. Therefore I and Q are also equal. Therefore ABC is bisected. *Q. E. D.*

[a] Per 2. *l.* 6.
[b] Per 27. *l.* 1.
[c] Per 5. *l.* 1.

PROP. IV. Theorem.

TRiangles which are equiangular to one another are like or similar, that is, have their Sides also (a) *that are opposite to the equal Angles proportional.*

(a) Per def. 1. *l.* 6.

In the Triangles X, Z, let the Angle A be equal to the Angle F, and the Angle C to the Angle L, and the Angle B to the Angle I; I say, that AB is to FI as AC is to FL; and AC is to FL as CB is to LI; and CB is to LI as BA is to FI. *Fig.* 7.

Demonst. If the Angle F be laid upon its equal A, the Sides FI, FL will fall upon the Sides AB, AC. And because the external Angle AIL is by the Hypothesis equal to the internal B (*d*), therefore (*e*) IL, BC, are parallel. Therefore BI is to IA (*f*) as CL to LA. therefore by compounding BA is to IF as CA to LF. And if the Angle L be laid upon the Angle C, it will be shew'd in the same manner, that AC is to FL, as BC is to IL; and if the Angle I be laid upon the Angle B, it will be shew'd in the same manner, that BC is to IL as AB to FI. The Proposition therefore is prov'd. *Fig.* 7, 8.

(d) *Fig.* 8, alone.
[e] Per 29. *l.* 1.
[f] Per 2. *l.* 6.

Corollaries.

1. IF in a Triangle a Line LI be drawn parallel to one Side BC, the Triangle LFI will be like to the whole CBF; and consequently CF will be to LF, as BC to LI. *Fig.* 8.

For because LI, BC, are parallel, the external Angles FIL, FLI will (*per* 27. *l.* 1.) be equal to the internal ones B and C: But F is common to both Triangles.

Therefore they are equiangular. Therefore the Sides CF, LF opposite to the equal Angles B and FIL (*f*) are proportional to the Sides BC, LI, which are oppos'd to the common Angle F.

(f) By the foregoing.

Fig. 9. 2. If in a Triangle a right Line BF drawn from the opposite Angle B, doth cut the Parallels AC, LO, it cuts them proportionally.

For by Coroll. 1. AF is to LI, as FB is to IB; and FC also is to IO, as FB is to IB. Therefore AF is to LI, as FC to IO. Therefore by changing AF is to FC, as LI to IO.

Fig. 51. [3. *From Coroll.* 1. *we learn to find the Heighth of a Tower, or any elevated Point, by only the Shadow of a Staff. Fix the Staff FL perpendicularly upon the Ground in that Place where the Ray of the Sun XBA, that terminates the Shadow of the Tower BZ, may also pass through L. There will be in the Triangle AZB the Line FL parallel to the Heighth of the Tower ZB. Whence as AF the Distance of the Staff from the Point of the Shadow, is to FL the Length of the Staff; so is AZ the Distance of the Tower from the Point of the Shadow, to ZB the Heighth of the Tower. And because the three first Terms are easily had by measuring, the fourth also*, i. e. *the Heighth of the Tower is had also.* Q. E. I.

Fig. 52. 4. *From this also incomparably useful Proposition, we may deduce that Famous Theorem of* Ptolomy; *to wit, That in every Quadrilateral inscrib'd in a Circle, the Rectangle of the Diagonals AC×BD is equal to the two Rectangles of the opposite Sides, AB×CD and AD×BC. For let the Angle BAE be made equal to the Angle CAD. Because the Angles BAE, CAD, are equal by Construction, the Angles ABE, ACD, standing upon the same Arch AD, are* * *equal; therefore the Triangles BAE, CAD, are alike. And AC: CD :: AB : BE; and consequently* † *the Rectangle of the Extremes AC×BE is equal to the Rectangle of the Means CD×AB. In like manner, because the Angle EAD is equal to BAC by Construction, and the Angles ADE, ACB, as standing upon the same Arch AB, are equal: The Triangles ADE, ACB, will be like; and AD : DE :: AC : CB. And therefore the Rectangle of the Extremes AD×CB is equal to the Rectangle of the Means DE×AC. But the Rectangles AC×BE, and AC×DE, are equal to the Rectangle AC×BD. Therefore the Rectangles AB×DC, and AD×BC,*

* Per 21. l. 3.

† Per 16. l. 6.

AD×BC, which are made by the opposite Sides, are equal to the Rectangle AC×BD, which is made by the Diagonals. Q. E. D.]

PROP. V. Theorem.

IF two Triangles have all their Sides proportional each to each, they shall also be mutually equiangular. Fig. 10.

That is, If AB be to RF, as AC to RQ; and as AC is to RQ, so is CB to QF; and as CB is to QF, so is AB to RF; I say, that the Angles opposite to the Antecedents, are equal to the Angles opposite to the Consequents; to wit, C to I, and B to F, and A to O.

Ang.	Antec.	Conseq.	Ang.
C	AB	RF	I
B	AC	RQ	F
A	CB	QF	O.

Make X and Z equal to A and C; and let the Sides meet in N. The Angles B and N will (*a*) be also equal. Because therefore the Triangles P, T, are equiangular, AB (by the foregoing) will be to RN, as AC to RQ. But by the Hypothesis, AB is to RF as AC to RQ. Therefore AB is to RF as the same AB is to RN. Therefore RN, RF are equal. In the like manner I might shew that QN and QF are equal. Therefore the Triangles T, S, are equilateral to each other. Therefore the Angles I, F, O, are equal (*per* 8. *l.* 1.) to the Angles Z, N, X, that is, by the Construction to the Angles C, B, A. *Q. E. D.* (a) *Per Corol.* 9. *p.* 32. *l.* 1.

PROP. VI. Problem.

IF two Triangles (P, S) have one Angle (A) equal to one Angle (O); and the Sides (AB, AC, RF, RQ) which contain the equal Angles proportional; the Triangles will be like.

Let X and Z be made equal to the Angles A, C, and the Sides meet together in N. Therefore the Angles B and N will (†) be alſo equal. Then it may be ſhew'd, as in the foregoing, that R F, R N, are equal. But R Q is common to both Triangles S, T. The Angles alſo O and X are equal, becauſe they are both equal to the ſame A, the one X by the Conſtruction, & O by the Hypotheſis. Therefore (*c*) I and F are likewiſe equal to Z and N. Therefore the Triangle S is equiangular to the Triangle T; that is, by the Conſtruction to the Triangle P. Therefore S, P, are like (*per* 4. *l.* 6.) *Q. E. D.*

† *Per Corol.* 9. *l.* 32. *l.* 1.

[c] *Per* 4. *l.* 1.

PROP. VII.

IS *ſcarce of any Uſe.*

PROP. VIII. Theorem.

Fig. 11.

IN *a rectangle Triangle, the Perpendicular* (BC) *let down from the right Angle to the Baſe, cuts the Triangles into Parts like to the whole, and betwixt themſelves.*

In the Triangles A B F and L, the Angle F is common, but the Angles A B F and X are by the Hypotheſis right ones, and conſequently equal. Therefore the other Angles A and O are (*e*) alſo equal. Therefore (*f*) the Triangles A B F and L are like. In the ſame manner the Triangles A B F and R may be ſhew'd to be equal, and the Angle I equal to the Angle F. From which it is now manifeſt, that R and L alſo are like, ſeeing the Angles I and F; O and A; U and X are equal. *Q. E. D.*

[e] *Per Corol.* 9. *p.* 42. *l.* 1.

(f) *Per* 4. *l.* 6.

Corollaries.

FIRST, BC is a mean Proportional betwixt AC, CF.

For

For seeing there be in the Triangle R and L,

equal Ang. I. F	equ. Ang. A. O
Sides oppos. A C. C B	Sid. oppos. C B. C F.

It is manifest (*a*) that A C : C B :: C B : C F. (a) *Per* 4. *l.* 6.

2. B F is a mean proportional betwixt A F, and C F. Likewise A B a mean betwixt F A and C A.

For in Triangle A B F and L.

equal Ang. A B F. X	equ. Ang. A. O
Sid. oppos. A F. B F	Sid. oppos. B F. C F

Therefore A F (*b*) : B F :: B F : C F. Likewise because in Triangle A B F and R there be (b) By the same.

equal Ang. A B F. V	equ. Ang. F. I
Sides oppos. A F. A B	Sid. oppos. AB. AC

It will be again A F : A B :: A B : A C.

3. *Hence we learn to measure an inaccessible Line, one Term whereof is accessible. Let the inaccessible Line be C F. Let there be rais'd from the Point C the Perpendicular C B: And to any Point of this Perpendicular as B, let there be applied a Square or any right Angle ABF; so that in looking along the Line B F the Point F, and along the Side B A the Point A may be observed. Let the accessible Line A C be measured, and from the following Analogy the inaccessible C F will be made known. AC : CB :: CB : CF. Let the Square then of the Line C B be divided by the Line A C, and the Quotient (c) will give the sought Line C F.* Q. E. I. Fig. 11. [c] *Per Corol.* 3. *p.* 17. *l.* 6.

PROP. IX. Problem.

TO *divide a given Line (A B) according to a given Proportion (F I to I L).* Fig. 12.

Let the infinite Line A Z be drawn. From which take A Q, Q R, equal to F I, I L. From R draw R B. Parallel to this draw Q C from Q. I say the thing is done.

It is manifest from *Prop.* 2. *l.* 6.

PROP. X. Problem.

Fig. 13. *To divide a given Line as (AB) in like manner as another given one (AI) hath been divided (in F, C).*

Let the right Line IB join the Extremities of the two Lines. Draw Parallels to this from the Points F, C, which may meet the right Line, that is to be cut, AB in L and Q. I say the Thing is done.

This is manifest from the Corollary of *Prop.* 2. *l.* 6.

Fig. 53. [*Or thus, if the cut Line IA be greater than that which is to be cut BQ, let three Circles touching one another be describ'd with the Diameters IF, IC, IA; and let the Subtense BQ be fitted from the Point I to the Circumference of the greatest Circle: The two lesser Circles will cut the Line BQ in the Points L, P, in the Proportion * of the Sections of the Diameter LA. If the Line IA be cut into four Parts, four Circles are to be drawn; if into five, then five Circles; and so infinitely.*]

* Per Corol. 4. p. 31. l. 3.

Scholium.

Fig. 13. FROM this Proposition we learn to cut a right Line given into any equal Parts whatsoever. Let an infinite right Line make any Angle with the right Line to be cut AB; from which take with a Pair of Compasses so many equal Parts AC, CF, FI, as you would divide AB into. Draw the right Line IB, and the Parallels to it FL, CQ. I say the Thing is done.

Fig. 14. We may do the same Thing otherwise, and more easily after *Maurolycus*, in the manner following. Let AB be to be trisected or divided into three equal Parts. Draw the infinite Line IX parallel to AB, above or below it. From IX, if it be below AB, take with a Pair of Compasses three equal Parts IQ, QR, RS, which together may be greater than AB; but lesser if IX is above. Through I and A, as likewise through S and B draw right Lines which may meet together in C. From C to Q and R draw right Lines: These will trisect the given Line AB. The Demonstration appears from *Coroll.* 2. *Prop.* 4.

Again,

Again, with *Maurolycus* we may otherwise obtain the same thing, to wit, thus: Let AB be to be quadrisected. Draw the infinite Line AX and BZ also an infinite Line parallel to it. From these take with the Compasses equal Parts AL, LO, OQ, and BV, VS, SR, in each fewer Parts by one, than are required in AB; then let there be drawn the right Lines, LR, OS, QV. These will quadrisect the given AB. Fig. 15.

For because by Construction, the Lines LO, RS, parallel and equal, are join'd by LR and OS, these also (*a*) will be parallel. In the like manner OS and QV are parallel. Therefore seeing AQ is cut into three equal Parts, AI will also (*b*) be cut into so many equal Parts. Likewise BC will be cut into three equal Parts. Therefore the whole AB will be cut into four equal Parts.

(a) Per 33. *l.* 1.
[b] Per Corol. *prop.* 2. *l.* 6.

These two Ways of Practice are easier than *Euclid*'s, because fewer Parallels are to be drawn.

PROP. XI. Problem.

TO find a third Proportional to two right Lines given (AB, BC). Fig. 16.

Draw the right Line AC. From BA produced take AF equal to BC. Through F draw the infinite Line FX parallel to AC, which infinite let BC produc'd meet in L. I say that AB is to BC, as BC to CL.

For AB : AF (*a*) :: BC : CL. But AF (*b*) is equal to BC. Therefore AB : BC :: BC : CL; and so CL is the third Proportional sought.

[a] Per 2. *l.* 6.
(b) By the Construction

Otherwise.

LET AB and BC be set at a right Angle. Join AC. From C draw CX perpendicular to AC infinite; which CX let AB produc'd meet in L. I say AB : BC :: BC : BL. It is manifest from Coroll. 1. pr. 8. Fig. 17.

Scho.

Scholium.

A Given Proportion may not only be continued in three, but also in infinite Terms, and the whole Sum of the infinite proportional Terms be exhibited. *Gregory* of St. *Vincent* hath very handsomely prosecuted this Matter, and the whole Business of Geometrical Progression in the whole second Book of his Work. We for the Sake of the Studious, will here present succinctly the Construction and Demonstration of the Thing proposed.

Problem.

Fig. 19. LET a Proportion of the greater Inequality be given, as AB to BC. It is requir'd to continue this thro' infinite Terms, and to present the Sum of them all.

Let the Perpendiculars AL, BO, be erected, and taken equal to the given Lines AB, BC, and through L, O, let a right Line be drawn, meeting with ABC produc'd in Z. I say, 1. If from C you erect the Perpendicular CQ; CQ shall be a third Proportional. Transfer QC into CE, and from E erect ER; this shall be a fourth Proportional. Transfer ER into EF, and erect FS; this shall be a fifth Proportional; and so the Proportion of AB to BC, that is, of AL to BO, will be continued through the Terms, AL, BO, CQ, ER, FS, *&c.* or AB, BC, CE, EF, FI, *&c.* infinitely, because every Term (as FS) may be taken away from the remaining one FZ; for seeing LA (that is, AB) is less than AZ; FS also (*a*) must ever be less than FZ.

(a) *Per Coral.* 1. *prop.* 4. *l.* 6.

I say, 2. AZ is equal to the whole Sum of the infinite Proportionals.

Part I. [*It being suppos'd as before, AZ : BZ :: AB : BC; it will be by alternating AZ : AB :: BZ : BC. And by dividing, AZ—AB : AB :: BZ—BC; that is, BZ : AB :: CZ : BC. Therefore by inverting AB : BZ :: BC : CZ. And by compounding AB+BZ : BZ : BC +CZ*

+CZ : CZ ; *that is,* AZ : BZ :: BZ : CZ.] But as AZ is to BZ, so is LA to OB; and as BZ is to CZ so is OB to QC. Therefore also LA is to OB, as OB is to QC. In the same manner I might shew that OB is to QC, as QC to RE; and so forwards infinitely.

Part 2. The whole Sum of the infinite Terms is neither less than AZ, nor greater; therefore it is equal. It is not greater, because seeing we have shew'd above, that QC is lesser than CZ, and RE than EZ, and SF than FZ, and so on infinitely, all the Terms QC, RE, SF, *&c.* may be infinitely set one by another in the right Line AZ, so that the Point Z shall never be reach'd. Again, the said Sum will not be less, because I have above shew'd, AZ, BZ, CZ, to be continually proportional; and in the same manner the same Thing is shew'd of the rest EZ, FZ, *&c.* Seeing therefore by transferring the Proportionals QC, ER, FS, *&c.* into CE, EF, FI, the Remainders EZ, FZ, IZ, *&c.* are always continually proportional, as we have already shewed; we shall at the last come unto a Remainder less than any given one; and therefore the Sum of the Proportionals shall exceed every Quantity that is less than AZ; from whence it self cannot be less than AZ. Seeing therefore it is neither greater nor less than AZ, it shall be equal to it. *Q. E. D.*

Theorem.

THE Difference of the first Terms, the first Term, aud the whole Sum of the infinite Proportionals, are continually proportional.

In the upper Figure let OX be drawn parallel to AZ. Therefore LX shall be the Difference of the first Term AL or AB, and of the second BO, or BC. Because XO is parallel to AZ; LX shall be to XO, as (*a*) LA is to AZ. But XO is AB, and LA likewise is AB. Therefore the Difference LX is to the first Term AB, as AB the first Term is to AZ the whole Sum. *Q. E. D.* *Fig.* 19. (a) *Per Corol.* 1. *prop.* 4. *l.* 6.

The same Thing may be demonstrated universally and very briefly in every kind of Quantity, thus; Let there be any continually Proportionals whatsoever (as well Numbers, as other Quantities) AZ, BZ, CZ, *&c.* and *Fig.* 20.

let

let them all be transferr'd upon the first AZ. Therefore AB, BC, CE, EF, *&c.* will be the Differences of the Proportionals; which together with the last Quantity IZ are equal to the first AZ. Now because if Proportionals be continued infinitely, the last Quantity vanisheth away, it is manifest that the Differences of the infinite Proportionals are equal to the first AZ. Then because AZ is to BZ, as BZ is to CZ, and so on. By dividing, AB will be to BZ, as BC to CZ, and by converting, as AB, the first Difference, is to AZ, the first Quantity; so BC, the second Difference, is to BZ, the second Quantity, and so forwards. Therefore as AB, the first Difference, is to AZ the first Quantity. So all the Differences, (that is, as I have already shew'd, the first Quantity AZ) are to all the Quantities, that is, to the whole Sum of the infinite Quantities. *Q. E. D.*

PROP. XII. Problem.

Fig. 21. *THree right Lines being given (AB, BC, AF) to find a fourth Proportional.*

Let the two right Lines be disposed, as the Figure shews, and draw the right Line BF, to which let the infinite Line CZ be made parallel. Let AF produc'd to L meet CZ.

I say, AB is to BC, as AF to FL, as is manifest from *Prop.* 2. of this Book. Therefore FL is the fourth Proportional sought.

Scholium.

OUR Countryman *Bettin* in his Treasury of Mathematical Philosophy, doth handsomely from 35. l. 3. and 14. of this, which depends not upon the present Proposition, find out a fourth Proportional, three being given, and a third two being given, after this manner.

Fig. 22. If three right Lines be given, let the second CB, and the third BD be join'd right to one another, so as to make one right Line, and let the first BA touch them in

in the Point B in what Angle you will. Through the Points C, A, D, describe a Circle (*a*), which let AB the first Line meet in the Point Z. BZ is a fourth Proportional. (a) *Per* 5. *l.* 4.

For seeing the Rectangles ABZ, CBD, are (*b*) equal, AB will be to BC, as BD to BZ, by the 14th of this Book, which, as was said, depends not upon this. (b) *Per* 35. *l.* 3.

If there be given two right Lines AB, BC; let BD equal to BC be join'd to BC, so as to make one strait Line. Then let the first AB touch BC in B in any Angle. Then the rest is as before, and BZ will be the third Proportional sought. *Fig.* 23.

The Demonstration is the same; for seeing the Rectangles ABZ, CBD, are equal, AB will be to BC, as BD (that is, BC) is to BZ.

PROP. XIII. Problem.

TWO right Lines given (AC, CB) to find a mean Proportional. *Fig.* 24.

Let the whole compound Line AB be bisected in O, and from the Center O a Circle be described through A and B; from C erect a Perpendicular CF, meeting the Circumference in F.

I say, AC is to CF, as CF is to CB.

For let AF, BF be drawn; the Triangle (*c*) AFB is right-angled, and from the right Angle there is drawn the Perpendicular FC to the Base. Therefore AC is to CF as (*d*) CF is to CB. (c) *Per* 31. *l.* 3. [b] *Per Coro l.* 1. *p.* 8. *l.* 6.

Corollary.

HEnce it is manifest, that if from any Point of the Circumference (as F) there be drawn a Perpendicular to the Diameter (FC), this Perpendicular is a mean proportional betwixt the Segments of the Diameter (AC, CB).

Scho-

Scholium.

THIS Place requires, that we should say something briefly concerning the finding out of two mean Proportionals betwixt two given Lines. All the Geometricians of *Greece*, at *Plato*'s Suggestion, set themselves with all their Might to the Solution of this Problem. Divers most subtle Ways of Practice are recited by *Eutocius* in his Commentary on *Archimedes*; as those of *Plato*, *Architas* the *Tarentine*, *Menæchmus*, *Eratosthenes*, *Philo Byzantius*, *Hero*, *Apollonius* of *Perga*, *Nicomedes*, *Diocles*, *Sporus*, *Pappus*; to whom the later Times have added *Verner*, *Gregory* of St. *Vincent*, *Renatus Cartesius*. Out of all these we shall select Three more easy than the rest.

Plato's *Method.*

Fig. 25. IT is requir'd to find out two Means betwixt the given Lines A B, B C.

Let A B, B C be set at a right Angle, and be produc'd infinitely towards X and Z. Then let two Squares (so our *Claudius Richards* hath it; for *Plato* himself made use of one Square only, but which had inserted into its Side * D E a Rule moveable along D E, let two Squares, I say) be taken, and the Angle D of one Square be applied to the right Line B X, in such certain wise, that one Side may also pass through A; and to the Point E in which the other Side cuts the right Line B Z, let a second Square be applied, which will pass through C. I say, that B D, B E, are two Means betwixt the given Lines A B, B C; that is, as A B is to B D, so is B D to B E, and B E to B C.

* See Fig. 26.

The Demonstration is manifest from *Coroll.* 1. *Prop.* 8. *l.* 6. for A D E is a right-angled Triangle, and from the right Angle to the Base there falls the Perpendicular D B. Therefore by the said Corollary, as A B is to B D, so is B D to B E; and for the same Cause, as B D to B E, so is B E to B C. Therefore betwixt the given right Lines A B, B C, there are found two mean Proportionals B D, B E. Which was the Thing to be done. This manner of solving the Problem is the easiest of all to be understood.

The

The Method of Phylo *the* Byzantine.

LET the two given right Lines A B, B C, be set together at a right Angle; then let the Rectangle A B C D be perfected, and let D A, D C be produc'd infinitely, and let the Diameters B D, A C be drawn, cutting each other in E. From the Center E through B let a Circle be drawn, which because A B C is a right Angle (*a*) will pass through A and C. Then let a Rule be applied to the Point B so, that the intercepted right Lines B G, O F, may be equal. I say, that A F, G C, are two mean Proportionals betwixt the given A B, B C; that is, as A B is to A F, so is A F to G C, and G C to C B. *Fig.* 27.

(a) *Per* 31. *l.* 3.

Demonst. Because G B, O F (*b*) are equal, O G, B F, will be also equal. Therefore the Rectangles O G B, B F O, that is (*c*) the Rectangles D G C, D F A, are equal. Therefore as G D is to D F, so (*d*) reciprocally A F is to G C, but G D is to D F (*e*) as B A to A F. Therefore as B A is to A F, so A F is to G C. Again, because I have already shew'd that A F is to G C, as B A is to A F; and since B A is to A F, as G D is to D F; that is, as G C is to C B, A F will also be to G C as G C is to C B. Therefore all four B A, A F, G C, C B, are continually proportional; and therefore betwixt the given Lines A B, B C two Means have been found. *Q. E. I.*

(b) By the Construction

(c) *Per Corol.* 1. *p.* 36. *l.* 3.

[d] *Per* 14. *l.* 6.

(e) *Per Corol.* 1. *p.* 4. *l.* 6.

These two Methods of Solution, although they be ingenious and easy enough; yet because a due Application of a Square and Rule is not made but by trying, they are not Geometrical.

The Method of Cartes.

LET an Instrument of such sort be provided; that two Rules may be open'd and shut about Y. Let there be inserted into these divers Squares connected together betwixt themselves in the Points B, C, D, E, F, G, in such sort that in the mean while that the Rules Y X and Y Z are open'd, the Square B C may impel the Square C D in the Rule Y Z, and the Square C D may impel *Fig.* 28.

impel the Square DE in the Rule YX, and the Square DE may impel EF, and EF impel or force forward FG and so on: But so that while the Rules YX and YZ are shut, all the Points B, C, D, E, F, G, tend to fall upon one and the same Point A. By this Instrument not only two, but also four and six, yea, as many Means as you will, betwixt two given right Lines may be found. Which thing can be obtain'd neither by the Sections of a Cone, nor by any Methods found out by the above-said Authors.

For two Means three Squares are requir'd, for four Means five Squares, and so on.

Let the lesser of the given right Lines be transferr'd upon the Rule YX, and let it be YB; the greater upon the Rule YZ, and let it be YE. Let the first Square be applied to the Point B, and be fixed there, and let the Rules be open'd, untill the Side of the third Square passeth through E. I say, that YC, YD, are two Means betwixt the given YB, YE; that is, that YB is to YC, as YC is to YD, and YD to YE.

The Demonstration appears out of *Coroll.* 2. *pr.* 8. *l.* 6. For from the Nature of the Instrument, in the Triangle YCD, the Angle at C is a right one, and from it CB falls perpendicular upon the Base YD. Therefore by the said Corollary, as YB is to YC, so is YC to YD. Again, because in the Triangle YDE, the Angle at D is a right one, and from it there falls the Perpendicular DC upon the Base YE, as YC is to YD, so is YD to YE. Therefore YB, YC, YD, YE are four continual Proportionals. Betwixt the given Line therefore YB, YE, there have been found two mean Proportionals YC, YD. *Q. E. I.*

If betwixt the given ones YB, YG, there be required four Means, open the Rules, untill the Side of the fifth Rule FG passeth through G. There will be YC, YD, YE, YF, four Means betwixt YB, YG. The Demonstration is manifest from the said Corollary.

This way, although the Instrument is more operose than *Plato*'s, is in very Deed an excellent one; both because it doth nothing by bare Tryal, and because it extends it self unto four and six, and as many Means as you will.

The Deliacal Problem, to wit, the Duplication of the Cube is performed by two Means, and all Bodies whatsoever are increas'd or diminish'd in a given Proportion

 (*a*) by

(*a*) by the same; like as the same Thing is perform'd in plain Figures (*b*) by one Mean. *Hippocrates* first open'd this way, which as the singular and only one, all Geometricians that have follow'd him have embrac'd.

(*a*) See *schol. pr.* 18. *l.* 12.
(b) *Corol.* 3. *pr.* 20. *l.* 6.

PROP. XIV. Theorem.

Equal Parallelograms (X, Z) which have one Angle (C) equal to one (O); have their Sides also, which are about the equal Angles, reciprocal; (that is, AC is to CB, as FO is to OL):

And if they have the Sides thus reciprocal, the Parallelograms are equal.

Fig. 29, 30.

Part I. Let IL and SB being produc'd meet together in Q. The Parallelogram X is to the Parallelogram R, as AC is to CB (*c*); and Z is to R (*d*), as FO to OL. But because by the Hypothesis X and Z are equal, X is to R as Z is to R. Therefore also AC is to CB as FO is to OL. *Q. E. D.*

[c] Per 1. *l.* 6.
(d) By the same.

Part II. As AC is to CB, so X is to R (*e*): And as FO is to OL, so is Z to R (*e*). But already by the Hypothesis AC is to CB, as FO to OL. Therefore X is to R as Z is to R. Therefore X and R are equal. *Q. E. D.*

(e) By the same.

[Coroll. *On this depends the Demonstration of the inverse Rule of Proportion. For in it there is always some Rectangle given as X; and one Side of another equal Rectangle, as CB; and the other Side is sought. As therefore AC the first Side of the given Rectan. is to CB, the given Side of the other Rectangle; so reciprocally FC the sought Side is to CL the second Side of the given Rectangle. The Rectangle therefore CB×FC is equal to the Rectangle AC×CL: And the latter Rectangle given being divided by the given Side of the former CB; the Quotient will give the sought Side FC.* Q. E. I.]

PROP. XV. Theorem.

Equal Triangles (ACL, FCB) which have one Angle (C) equal to one (O) have also their

Fig. 31, 32.

Sides about the equal Angles reciprocal (that is, AC is to CB, as FO to OL).

And if they have their Sides thus reciprocal, the Triangles are equal.

Let the right Line LB be drawn; the rest of the Demonstration is the same as that of the foregoing.

Corollary.

AS well Parallelograms as Triangles which have their Bases and Altitudes reciprocal are equal: And so conversely.

It is manifest from the two foregoing Propositions.

PROP. XVI. Theorem.

Fig. 33. *IF four right Lines (AB, FI; IL, BC) be proportional, (that is, if AB be to FI as IL is to BC) the Rectangle (X) under the Extremes (AB, BC) is equal to the Rectangle (Z) under the Means (FI, IL).*

And if a Rectangle under the Extremes be equal to a Rectangle under the Means, those four right Lines will be proportional.

Part I. In the Rectangles X and Z, about the right and therefore equal Angles B, I, by the Hypothesis AB is to FI as reciprocally IL to CB. Therefore X and Z (*a*) are equal. *Q. E. D.*

(a) *Per* 14. *l.* 6.

Part I. Because X and Z are now suppos'd equal; therefore (*b*) about the equal Angles B and I, AB is to FI as reciprocally IL to BC. *Q. E. D.*

[b] By the same.

[Coroll. (1.) *Hence it is easy to apply the given Rectangle Z (c) to the given right Line AB; to wit, by making AB : FI :: IL : BC. For BC is the Rectangle Z applied to the given right Line AB.*]

[c] Per 12. *l.* 6.

[Coroll. (2.) *Upon this Proposition depends the Demonstration of the direct Rule of Proportion. For in it there is always given some Rectangle as CL: And another like Rectangle is sought, one Side whereof is also given. It will there-*

fore be, as BC the first Side of the Rectangle given, is to EO the Side of the Rectangle sought; so directly CE, the second Side of the Rectangle given, is to OA the other sought Side. Therefore the Rectangle CE×EO is equal to the Rectangle BC×OA. And the Rectangle CE×EO being divided by BC the Quotient, will give OA the other Side which was sought. Q. E. I.]

PROP. XVII. Theorem.

IF three right Lines (AB, FL, BC) be proportional, the Rectangle under the Extremes (AB, BC) shall be equal to the Square of the Mean (FL). Fig. 34.

And if the Rectangle under the Extremes be equal to the Square of the Mean, those three right Lines are proportional.

Part I. Let O be taken equal to the Mean FL. Because therefore by the Hypothesis AB is to FL as FL to BC, and O is equal to FL; AB will also be to FL, as O is to BC. Therefore (*a*) the Rectangle under the Extremes AB, BC, is equal to the Rectangle under the Means FL and O, that is, is equal to the Square of FL.

[a] By the foregoing.

Part II. This is demonstrated in like manner from the second Part of the foregoing.

Corollary.

FRom this, taken together with the 13th, it is manifest, that if in a Circle FC be perpendicular to the Diameter, the Rectangle ACB is equal to the Square of FC. Fig. 24.

[(2.) *If A×B be equal to the Square of C; then A : C :: C : B.*

(3.) *If A : C :: C : B; and Cq be divided by A, the Quotient (b) will be B.*

[b] *Per Corol. 2. p. 16. l. 6.*

PROP. XVIII. Problem.

Fig. 35. *UPon a given right Line (RS) to describe a Polygon like, and in like manner posited to a given one (BQ).*

Resolve the given Polygon BQ into Triangles. Upon the given right Line RS make the Angles (*a*) R, O, equal to the Angles B, A. The Sides then will meet together in X. Upon XS make the Angles V, I, equal to the Angles T, C. The Sides will then meet together in Z. I say the Thing is done.

(a) Per 23. l. 1.

For because the Angles R, O, are equal to the Angles B, A, the Angles E, K, must also be equal (*per Coroll.* 9. *pr.* 32. *l.* 1.); and because also by the Construction V is equal to T, the whole EV must be equal to the whole KT. In like manner because O, I, are equal to A, C, respectively, the whole Angles OI, AC, must be equal. And because V and I also are equal to T & C by the Construction, Z and Q likewise must be equal (*per Corol.* 9, *pr.* 32. *l.* 1.) to T and C. Therefore the Polygons RZ, BQ, are mutually equiangular. It remains, that we shew that their Sides also are Proportional. RS is to BF * as SX to FL; and again, SX is to FL (*b*), as SZ to FQ. Therefore *ex æquo* RS is to SZ, as BF to FQ, *&c.*

* Per 4. l. 6.
[b] By the same.

Coroll. *Hence is derived the Method of making Maps or Charts, whether Geographical, or Chorographical, or those which Surveyors of Land make; and of framing Ichnographical Delineations of Fields, Buildings, Countries. For they are nothing else but the Reduction of great Figures unto like Figures which are of a small Compass, which is performed by the Means of this Proposition.*

PROP. XIX. Theorem.

Fig. 36, 37. *THE Proportion of like Triangles (X, Z) is duplicate of the Proportion of their Sides (AC, FI) which are subtended to the equal Angles.*

That is, if it be made * as AC is to FI, so is FI to a third AQ; the Triangle X is to Triangle Z, as AC the

* Per 11. l. 6.

the first is to the third Proportional AQ. See *Defin.* 10. 5.

Because the Triangles X, Z are like, BA, will be to LI (*c*) as AC is to IF. But by the Construction, as AC is to IF, so is IF to AQ. Therefore also BA is to LI, (*d*) as IF to AQ. Therefore in the Triangles QBA and Z, the Sides about the Angles A, I, which by the Definition of like Triangles are equal, are reciprocal. Therefore QBA and Z are equal (*e*). But the Triangle X is to QBA, as the Base AC to the Base AQ (*f*). Therefore X is to Z as AC to AQ. *Q. E. D.*

[c] Per 4. l. 6.

(d) Per 15. l. 6.

[e] Per 1. l. 6.

Coroll. *Hence is their Error to be corrected, who think that like Figures are in the same Proportion to one another, that their Sides are. For if of two, not only like Triangles, but also Squares, Pentagons, Hexagons, &c. yea, and Circles also, the Sides or Diameters be betwixt themselves as 2 to 1, the Figures or Area's themselves are as 4 to 1: If the Sides be betwixt themselves as 3 to 1, the Figures themselves or Area's are as 9 to 1; to wit, in a duplicate Proportion of those Sides.*

PROP. XX. Theorem.

LIKE Polygons (ABCDE, FGHIK) are divided, (1.) into like Triangles (P, S, and Q, T, and R, V) in Number equal: (2.) And proportional to the Wholes: And (3.) the Proportion of the Polygons is duplicate to that of the Sides, (AB, FG) which are betwixt the equal Angles (B, G, and BAE, GFK). Fig. 38.

Part I. Because the Polygons are like, they are mutually (*per Defin.* 1. *l.* 6.) equiangular, and their Angles equal BAE to GFK, and B to G, and BCD to GHI, and CDE to HIK, and E to K. Because therefore AB is to BC (*a*) as FG to GH, and the Angles B and G are equal, the Triangles P, S, (*b*) are like. In like manner it will be demonstrated that R and V are like. Then because the Wholes BCD, GHI, and the subducted ones BCA, GHF, are equal, the remaining ones also, ACD, FHI, are equal. In the same manner

(a) By the same.

(b) Per 6 l. 6.

I might shew that ADC, FIH, are equal. Therefore (*per Corol.* 9. *pr.* 32. *l.* 1.) the third CAD is equal to the third HFI. Where also (*e*) the Triangles Q and T are like. The first Part thereof is manifest.

[e] Per 4. l. 6.

Part II. Because P and S are like, the Proportion of P to S is duplicate to that of (*f*) CA to HF. But for the same Cause also the Proportion of Q to T is duplicate to the Proportion of CA to HF. Therefore P is to S as Q to T. In the same manner I will shew that as Q is to T, so R is to V. Therefore as one Antecedent P is to one Consequent S, so all the Antecedents P, Q, R, taken together, are to all the Consequents S, T, V, taken together, that is, so is Polygon to Polygon. Which was the other Part.

[f] By the foregoing.

Part III. The Proportion of P to S is duplicate (*h*) to that of AB to FG. But the Proportion of Polygon to Polygon is the same with the Proportion of P to S, as I have already shew'd. Therefore also the Proportion of Polygon to Polygon is duplicate to the Proportion of AB to GF. Which was third Part.

[h] By the foregoing.

Corollaries.

1. ALL ordinate or regular Figures, as Squares, equilateral Triangles, Pentagons, *&c.* are betwixt themselves in the duplicate Proportion of the Sides. For all regular Figures are like, as is manifest from *Defin.* 1. 6.

Fig. 38.

2. If in any like Figures whatsoever, the Sides AB, FG, which are placed betwixt equal Angles be known, the Proportion of the Figures is also known. As for Example, Let AB be of two Feet, and FG of six Feet, and as 2 is to 6, so let 6 be to some other Number; to wit, 18. The lesser Figure is to the greater as 2 is to 18, or as 1 is to 9. Now a third proportional Number is found, if (*per Corol.* 3. *pr.* 17. *l.* 6.) the second of the given ones be multiplied by it self, and the Product divided by the first.

Fig. 39.

3. From the same Proposition is drawn the excellent Method of increasing or diminishing any rectilineal Figure in a given Proportion. As if I would make a Pentagon, whose Side is AB fivefold of another. Find a Mean proportional BX (*i*) betwixt the Terms of the Pro-

(i) Per 13. l. 6.

Proportion given, A B, B C; upon this Frame (*a*) a Pentagon like to the given one. This shall be quintuple of the given one. [a] Per 28. l. 6.

For by the 20. the Pentagon A B is to B X, which is like to it, as A B the first is to B C the third Proportional.

Moreover, seeing the Proportion of Circles also is duplicate to the Proportion of their Diameters, as will be shew'd, p. 2. l. 12. This Practice belongs likewise to Circles.

[Schol. *Seeing the Proportion of the Squares E, K, is duplicate of the Proportion of their Sides O R, S V; from thence the duplicate Proportion of the Sides O R, S V is wont commonly to be expres'd by the Proportion of O R q to S V q.*] Fig. 41.

PROP. XXI. Theorem.

FIgures (A, B) which are like to the same (C) are also like betwixt themselves. Fig. 40.

This is manifest from *Defin.* 1. *l.* 6. and from *Axiom* 1. *l.* 1.

PROP. XXII. Theorem.

IF four or more right Lines (F I, L Q, and O R, S V,) be proportional; like Figures, and in like Sort described by them (A, B, and E, K,) must also be proportional. Fig. 40, 41.

And Conversly.

The Demonstration of the first Part is manifest. For because the Proportions of A to B and E to K are duplicate to the Proportions of F I to L Q, and O R to S V, which are by Hypothesis equal; themselves also must be equal.

The second Part is manifest also.

[Coroll. *If the right Line A B be cut in any manner in C; the Rectangle contain'd under the Parts A C, C B, is a Mean proportional betwixt their Squares. Likewise the Rect-* Fig. 24]

*angle contain'd under the whole AB, and one Part AC or CB is a Mean proportional betwixt the Square of the whole AB, and the Square of the said Part AC or CB. For (per Coroll. 1. p. 8. l. 6.) it is manifest that AC : CF :: CF : CB. Therefore AC Square : CF Square :: CF Square : CB Square. That is, * AC Square : Rectangle ACB :: Rectangle ACB : CB Square. Q. E. D.*

* Per 17. l. 6.

Moreover, (per Coroll. 2. p. 8. l. 6.) BA : AF :: AF : AC. Therefore BAq : AFq :: AFq : ACq. That is, † BAq : BAC Rectangle :: BAC Rectangle : ACq. In the same manner ABq : ABC :: ABC : BCq. Q. E. D.]

† Per 17. l. 6.

PROP. XXIII. Theorem.

Fig. 42.

Equiangled Parallelograms (X, Z) have betwixt themselves a Proportion that is compounded of the Proportions of their Sides (AC to CB, and LC to CF).

That is, if you make CB to be to O, as LC to CF, X is to Z, as AC is to O.

Let IL, SB, meet together in Q. The Parallelogram X (*a*) is to the Parallelogram R, as AC is to CB; and R is (*b*) to Z as LC is to CF, that is, as CB is to O. Therefore *ex æquo* X is to Z, as AC is to O. *Q. E. D.*

[a] Per 1. l. 6.
[b] By the same.

Corollaries.

Fig. 42.

FRom hence, and from 34. l. 1. it is manifest,

1. That Triangles which have one Angle (at C) equal, have that Proportion betwixt themselves, which is compounded of the Proportions of the right Lines AC to CB, and LC to CF. Which Lines contain the equal Angle.

2. That Rectangles, and consequently all Parallelograms whatsoever, have betwixt themselves the Proportion which is compounded of the Proportions of the Base to the Base, and the Heighth to the Heighth. And in the same manner we reason about Triangles.

3. Hence

3. Hence the Proportion of Triangles and Parallelograms may be readily learned. Let X and Z be the Parallelograms, and their Bases AC, CB, and CL, CF be their Heighths. Let it be made (c) as the Altitude CL, is to the Altitude CF, so is one of the Bases CB, to O. The Parallelogram X is to the Parallelogram Z, as AC to O. Fig. 42. (c) *Per* 12. *l.* 6.

PROP. XXIV. Theorem.

IN every Parallelogram (as SF) the Parallelograms which are about the Diameter (AB), to wit, (CL, OL) are both like to the whole Parallelogram, and to each other. Fig. 43.

By 27. 1. the Angles C, S, and L, F, are equal. By the same E is equal to I, that is, by the same, equal to A it self; but B is common both to the whole SF, and the Part CL. Therefore the whole SF, and the Part CL, are equiangular. It remains to be shew'd that they have the Sides opposite to the equal Angles proportional.

Because in the Triangles BCE, BSA, CE is parallel to SA, BC (by *Corol.* 1. *pr.* 4. *l.* 6.) will be to CE, as BS to SA: And CE will be to EB (by the same *Coroll.*) as SA to AB. But because in the Triangles ELB, AFB also, EL is parallel to AF, EB (by the same *Coroll.*) will be to EL, as AB to AF. Therefore *ex æquo* CE is to EL, as SA to AF. Therefore (by *Defin.* 1. *l.* 6.) CL and the whole CF are like. In the same manner, I might shew OI to be like to the whole SF. Therefore (*per* 21. *l.* 6.) CL and OI are also like betwixt themselves. *Q. E. D.*

PROP. XXV. Problem.

TO change a given Polygon (A) into another like to a given one (B). Fig. 46.

Or to make a Polygon equal to a given one, (A) and like to another given one (B).

Upon

Upon CF the Side the Polygon B a like one to which is required, (by 45. l. 1.) make a Rectangle Q equal to B. Then upon FS (by the same *Prop.*) make a Rectangle R equal to A. It is manifest that CF and BI do make one right Line. Betwixt CF and FI find a mean proportional FL (*a*). Upon this (*p.* 18. *l.* 6.) make a Polygon like to the given one B. this must also be equal to the given one A.

(a) *Per* 13. *l.* 6.

For seeing by the Construction, CF, FL, FI, are three Proportionals, the Polygon B is to the Polygon like to it which is made upon FL, as CF is to FI (*per* 20. *l.*6. and *Defin.* 10. *l.* 5.); that is, (*per* 1. *l.* 6.) as Q is to R. Therefore also by changing, as the Polygon B is to Q, so is the Polygon FL to R. But by the Construction the Polygon B is equal to Q. Therefore also the Polygon upon FL, which is like to B, is equal to R; that is, by the Construction to the given A. That therefore is done which was required.

PROP. XXVI. Theorem.

Fig. 44. *LIKE Parallelograms (BD, FN) having a common Angle (A) are about the same Diameter.*

Draw the right Lines AE, CE. If you deny that AEC is a common Diameter to the Parallelograms BD and FN; let another right Line AGC which cuts FE in G, be the Diameter of BD, and draw the Parallel GH. The Parallelograms FH, BD will be therefore about the common Diameter AGC, and consequently (by 24. l. 6.) will be like. Therefore (*per defin.* 1. *l.* 6.) as BA to AD, so is FA to AH. But also, as BA to AD, so is FA to AN, seeing BD, FN are like by the Hypothesis. Therefore FA is to AH, as the same FA is to AN. Which is absurd.

PROP. XXVII, XXVIII, XXIX.

THese cause Trouble to, and perplex Beginners, and are scarce of any Use.

PROP.

PROP. XXX. Problem.

TO cut a given right Line (AB) so that the whole (AB) shall be to one Segment (AC) as the same Segment is to the remainder (CB.) Fig. 45.

That is, as Geometricians speak, to cut a Line in extream and mean Proportion.

By 11. *l.* 2. so cut AB in C, that the Rectangle under AB, CB may be equal to the Square of AC. I say the Thing is done.

For by the 17th of this Book, as AB is to AC, so is AC to CB.

The Force of this Section of a Line is admirable in the inscribing and comparing regular Bodies.

PROP. XXXI. Theorem.

IF from the Sides of a rectangular Triangle (ACB) like Figures whatsoever be describ'd, that which is oppos'd to the right Angle, will be equal to the two others (L, R) taken together. Fig. 47.

Here *Prop.* 47. *l.* 1. is made universal.

From the right Angle C let the Perpendicular CO be let down. Because (*per Coroll.* 2. *p.* 8. *l.* 6.) AB, BC, BO are three Proportionals, F shall be to the Figure R, which is like to it, as AB the first, to BO the third Proportional, (to wit, by 20. *l.* 6. and *Defin.* 10. *l.* 5.) Again, because (by the aforesaid Corollary) BA, AC, AO are three Proportionals, the Figure F shall (by the foresaid *Prop.* and *Defin.*) be to L, which is like to it, as BA the first, to AO the third Proportional. Because therefore F is to R as AB is to BO; and the same F is to L as AB to AO; F shall also be to R and L taken together, as AB is to BO, AO taken together. But AB is equal to the two BO, AO. Therefore also F shall be equal to the two R and L. *Q. E. D.*

Coroll.

Coroll.

FRom this Proposition we can easily find one rectilineal Figure, equal and like to any Number of rectilineal Figures whatsoever, by the same Method, whereby *Probl.* 1. *Schol. pr.* 47. *l.* 1. one Square is found equal to any Number of given Squares whatsoever. Only in the Demonstration, let 31. *l.* 6. be cited instead of 47. *l.* 1.

Coroll. (2.) *A Circle upon the Hypotenuse of a Rectangle Triangle, is equal to two Circles describ'd upon the Sides. for all Circles are like amongst themselves; and are to one another as the Squares of their Diameters, by the second of the 12th Book.*

Fig. 54. Coroll. (3.) *From hence we may derive that Quadrature of Lunets (or little Moons) which* Hippocrates *of* Chios *first taught.*

For let ABC be a rectangle Triangle; and BAC a Semicircle to the Diameter BC: BNA a Semicircle describ'd on the Diameter AB: AMC a semicircle describ'd upon the Diameter AC. Thus therefore the Semicircle BAC is equal to the Semicircles BNA, and AMC together. If therefore you take away the two Spaces BA, AC, common on both Sides, there will be left the two Lunets BNA, AMC, bounded on both Sides with circular Lines equal to the rectilineal Triangle BAC. And if the Line BA be equal to the Line AC, and you let fall a Perpendicular unto the Hypotenuse BC, the Triangle BAO will be equal to the Lunet BNA, and the Triangle COA equal to the Lunet CMA. Q. E. I.

PROP. XXXII.

THIS is hardly of any Use, and hath nothing remarkable in it.

PROP.

PROP. XXXIII. Theorem.

IN the same or equal Circles, the Angles whether at the Centers (as ABC, FOD); or at the Circumference (as ARC, FSD) have that Proportion betwixt themselves, which the Arches (AKC, FGD) on which they stand have. Understand the same Thing of Sectors. Fig. 48.

As for the Angles at the Center and the Sectors, it will be demonstrated altogether in the same manner, in which *Prop.* 1. of this Book it was demonstrated, that Triangles of the same Heighth are as their Bases. Only where *Prop.* 38. *l.* 1. is cited there, let *Prop.* 29. *l.* 3. be cited here.

And because the Angles R and S at the Circumference are Halves of the Angles ABC, FOD, at the Center, that which hath been demonstrated of these will be manifest also of those.

Coroll.

1. THE Angle (BAC) at the Center, is to four right Angles, as the Arch B on which it stands, is to the whole Circumference. Fig. 49.

For as BAC is to the right Angle BAF, so by this 33. the Arch BC is to the Quadrant BF. Therefore the Angle BAC is to four right Angles, as the Arch BC is to four Quadrants, that is, the whole Circumference.

2. The Arches IL, BC of unequal Circles, which do subtend equal Angles, whether at the Center, as IAL and BAC, or at the Circumference, are like Arches.

For the Arch IL is (by *Coroll.* 1.) to its Circumference as the Angle IAL, that is, BAC is to four right Angles; and the Arch BC is to its Circumference (by the same *Corollary*) as the same Angle BAC is to four right

right ones. Therefore IL is to its Circumference, as BC is to its. Therefore (by *Defin.* 4. *l.* 6.) the Arches IL and BC are like.

3. Two Semidiameters (AB, AC) do take away from concentrical Circumferences like Arches IL, BC. This is manifest from *Coroll.* 2.

4. The Segments (BKC, IOL,) which contain equal Angles (K, O,) are like.

For by *Coroll.* 2. the Arches BC, IL, and consequently the Arches BKC, IOL are like.

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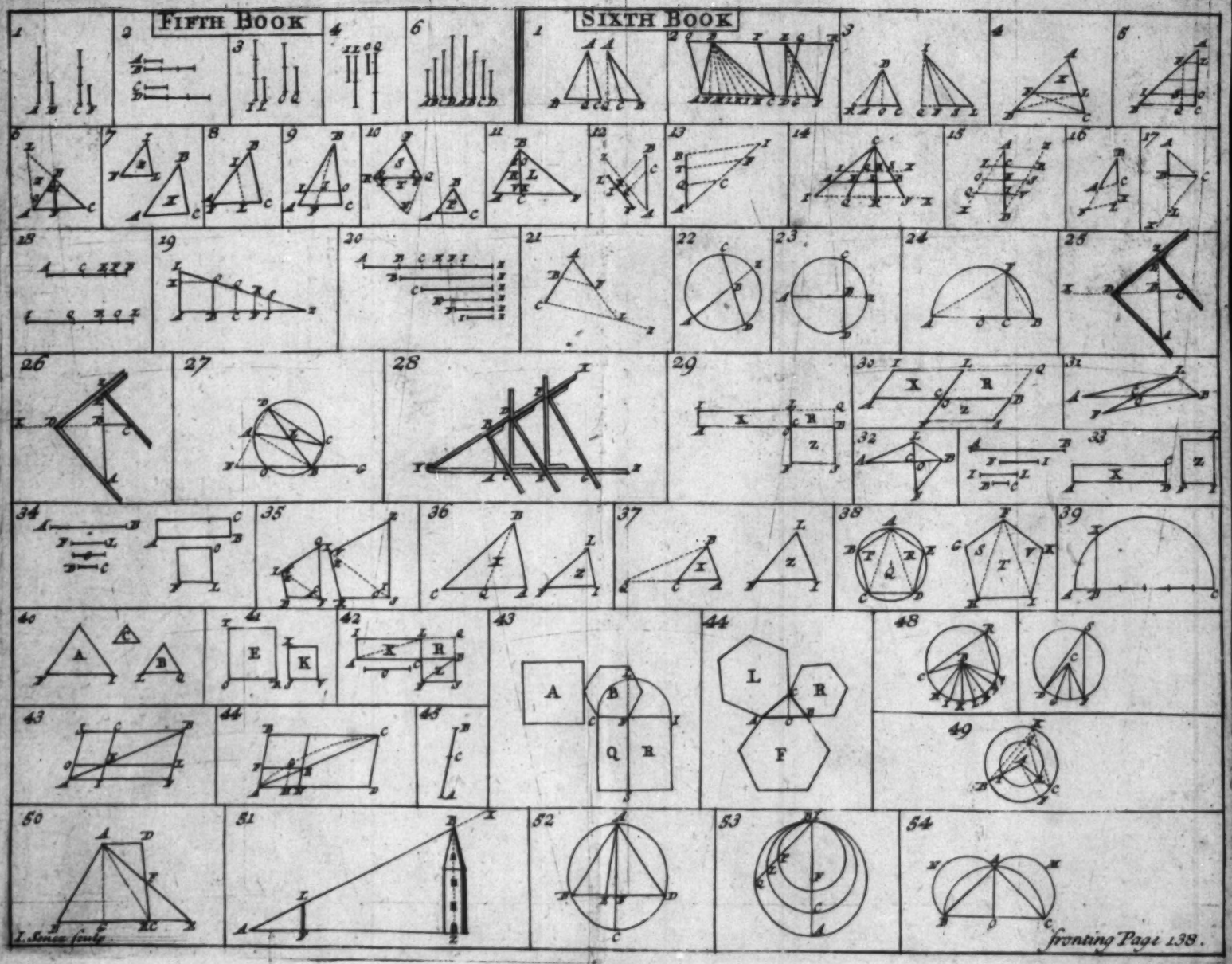

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fronting Page 138.

The Elements of *Euclid.*

BOOK XI.

With Us the SEVENTH.

TO the six first Books *Euclid* subjoins the Elements of Numbers comprehended in the three following, the Seventh, Eighth, and Ninth, to which he also adjoins a Tenth, concerning incommensurable Quantities. We pass immediately from Plains to Solids; purposing to treat of Numbers separately. Seeing it will, I suppose, be more commodious for Learners, if the Elements of Geometry be not interrupted, by treating of any other Matter, but be had all together. Nevertheless when we shall cite the Propositions of This and the following Book, we shall not call these Books the Seventh, and the Eighth, but the Eleventh, and the Twelfth, lest if we should depart from the every where received Order of *Euclid*, the Citation of Propositions should thereby be rendred more intricate.

This Book in a sort contains two Parts. In the first are laid the Foundations, on which the whole Doctrine of Solids; that is, of Bodies depends. In the other the Affections of Parallelipipeds are propounded.

This Eleventh Book of Elements sets forth the first Principles of Solids. Nor can indeed the Properties of Bodies be known without it; and if we set upon almost any Part of the Mathematicks, without the Knowledge of Solids, we shall labour in vain, or be at least at a great Loss. For the Spherical Doctrine of Theodosius, *Spherical Trigonometry also, a great Part of practical Geometry, Staticks, and Geography, depend upon it, and what Things occur of any great Difficulty*

in

in the Art of Dialling, in the Conic Sections, Astronomy, Dioptricks or Opticks, do all become more easy, the Principles of Solids being once understood. So that those who have delivered the Elements of Geometry, leaving out and setting aside this and the following Book, are to be reckon'd to have delivered the same very imperfectly.

DEFINITIONS.

1. A Solid or Body is that which hath Length, Breadth, and Thickness.

2. The Extreme of a Solid is a Surface.

Fig. 1. l. 11. 3. The right Line (A B) is to the Plain (C C) right or perpendicular, when it makes right Angles (B A C, B A C) with all the right Lines (C A) in the Plain (C C) by which it is touch'd.

Fig. 2. 4. A Plain is right or perpendicular to a Plain, when all the right Lines (L Q) which are drawn in one of the Plains perpendicular to the common Section (X R) are right or perpendicular to the other Plain (A B C O).

Fig. 3. 5. If the right Line (O L) stands upon a Plain not at right Angles, and from its highest Point (L) there be drawn to the Plain the Perpendicular (L P) and (O P) be join'd; the Angle (L O P) is said to be the Inclination of the Line (O L) to the Plain.

Fig. 4. 6. If the Plain (R E) doth not stand perpendicularly upon the Plain (L Q) the Inclination of one to the other is the acute Angle (A B C) which is contain'd by the right Lines (A B and B C) which are drawn in both Plains perpendicular to the common Section (O E).

7. A Plain is said to be alike inclin'd to a Plain, as is some other Plain to another, when the said Angles of their Inclinations are equal.

8. Parallel Plains, are those which being continued every way, are always distant from each other by equal Intervals.

9. Like solid Rectilineal Figures are those which are contain'd under like Plains, in Number equal.

Fig. 5. 10. A solid right-lin'd Angle is that which is contain'd under plain Angles more than two (B A C, C A C, O A B) which are not in the same Plain, meeting together in one Point.

11. Equal solid Angles are those, which being conceiv'd to be put each within the other, do agree or perfectly coincide.

Like as a plain Angle is a mutual Inclination of Lines, so a solid Angle is an Inclination of Surfaces. Concerning both therefore we must reason in the same manner.

12. A Prism is a solid Figure, comprehended by Plains, amongst which two opposite ones (OFE, ACB) are parallel, equal, and like. *Fig. 6, 7, 8.*

13. A Parallelopiped is a Solid contain'd under quadrilateral Plains, of which the opposites are parallel. *Fig. 8.*

14. If six Plains in which the Opposites are parallel be Squares, the Solid contain'd by them will be a Cube.

PROPOSITION I. Theorem.

ONE Part (AC) of a right Line cannot be in a Plain (OE); and another part (CB) out of it. *Fig. 9.*

It is clear of it self from the Definition of a Plain and a right Line. See *Defin.* 7. and 4. *l.* 1.

PROP. II. Theorem.

EVery Triangle is in one Plain. And two right Lines cutting each other, are in the same Plain. *Fig. 10.*

For if a Plain be applied to one of its Sides, and to the Point of meeting of the other two, it will be evident that the whole Triangle is in that Plain.

PROP. III. Theorem.

IF two Plains (AB, CD) cut each other, (EF) their common Section is a right Line. *Fig. 11.*

It is manifest from the Definition of a Plain.

But we may demonstrate it thus. If E F the common Section be not a right Line, let there be drawn in the Plain C D the right Line E O F, and in the Plain A B the right Line E Q F. The two right Lines therefore E O F, E Q F, will inclose a Space. Which is absurd.

PROP. IV. Theorem.

Fig. 12. *IF a right Line (BA) be perpendicular to two right Lines (CAX, FAS) which cut each other, it will also be perpendicular to the Plain which is drawn through them.*

If you deny it, let another right Line B Q be perpendicular to the Plain of the right Lines A C, A F. Join A Q, and to this in the Plain F A C draw the Perpendicular Q O. This being produced will necessarily cut (as is gathered from *Schol. Prop.* 31. *l.* 1.) one of the right Lines C A X, F A S, or both, wheresoever the Point Q shall be. Therefore let it cut C A X in O, and let B O be join'd. Because therefore the Angle B A O is by the Hypothesis a right one;

The Square of B O shall be equal to B A Squ. + A O Squ. (b).

(b) *Per* 47. *l.* 1.

But because B Q is suppos'd perpendicular to the Plain F A C, and consequently (by *Defin.* 3. *l.* 11.) makes a right Angle with A Q;

B A Squ. is equal to B Q Squ. + A Q (d).

(d) *Per* 47. *l.* 1.

And because the Angle A Q O is by the Construction a right one;

A O Squ. is equal to O Q Squ. + A Q Squ. (e).

(e) By the same.

Therefore B O Squ. is equal to B Q Squ. + O Q Squ. + A Q Squ. twice taken.

There-

Therefore BO Square is greater than the Squares of BQ and OQ; and (as is clear from *Prop.* 47. *l.* 1.) consequently BQO is not a right Angle. Therefore BQ is not perpendicular to the Plain (by *Defin.* 3. *l.* 11.) CAF. Therefore the Proposition is manifest.

Scholium.

FRom its being suppos'd that BQ is perpendicular to the Plain FAC; it is directly demonstrated that it is not perpendicular to that Plain; and consequently from the denial of the Assertion of the Theorem, the same Assertion is directly proved. This Demonstration as to the Substance of it is *John Cierman*'s.

PROP. V. Theorem.

IF three right Lines (BA, CA, FA) be perpendicular to the same right Line (AR) at the same Point (A); those three will be in one Plain. Fig. 13.

For, if it may be, let one of them BA be in another Plain (RO) which may cut LQ the Plain of the other two CA, FA, in the right Line AO. Because by the Hypothesis RA stands perpendicularly upon the two CA, FA, it will be perpendicular to the Plain LQ (by the foregoing) Therefore RA makes a right Angle with AO (by *Defin.* 3. *l.* 11.) But also by the Hypothesis RAB is a right Angle. Therefore the Angles RAB and RAO are equal. Which is absurd.

PROP. VI. Theorem.

RIGHT Lines (AB, CD) which are perpendicular to the same Plain (CF) are parallel. Fig. 14.

It might be taken for granted as a Thing of it self known; but we may demonstrate it thus.

BD being join'd, make in the Plain FE the Line DG perpendicular to BD, and equal to BA; and let DA, GA, GB, be join'd. The right Lines BD, DG, are equal to BD (*a*) and BA; and the Angles BDG, (*b*) DBA are right ones. Therefore (*per.* 4. *l.* 1.) AD, BG, are equal. Therefore the Triangles ABG, GDA are equilateral to each other, and consequently the Angles ABG, ADG are equal. But ABG (by *Defin.* 3. *l.* 11.) is a right Angle. Wherefore ADG is also a right one. But BDG also by the Construction, and CDG by *Defin.* 3. are right Angles. Therefore GD is perpendicular to the three Angles CD, AD, BD. Therefore CD is (*c*) in one Plain with AD, and BD. But AB also is in one Plain (*per* 2. *l.* 11.) with AD, and BD. Therefore AB, CD are in one Plain. Therefore seeing the Angles ABD, CDB (by *Defin.* 3. *l.* 11.) are right ones, AB, CD will (*per* 29. *l.* 1. and *Defin.* 36. *l.* 1.) be parallel Lines. *Q. E. D.*

(a) By the Construction
[b] Per def. 3. *l.* 11.
(c) By the foregoing.

PROP. VII. Theorem.

Fig. 15.

A Right Line (EF) cutting right Lines (AB, CD) placed in the same Plain, is in one and the same Plain with them.

It might be taken for granted. But he that will may thus demonstrate it.

Let another Plain cut the Plain of the right Lines AB, CD, in the Points E, F. If now EF is not in the Plain of AB, CD, it is not the common Section. Let EGF therefore be so. Therefore (*per* 3. *l.* 11.) EGF is a right Line; the two right Lines therefore EF, EGF inclose a Space. Which is absurd.

Corollary.

HEnce it follows, that if EF cut the Parallels AB, CD, it is in the same Plain with them. For (by *Defin.* 36. *l.* 1.) any two Parallels are in the same Plain.

PROP.

PROP. VIII. Theorem.

IF of two Parallels (AB, CD) one (AB) be perpendicular to a Plain (EF); the other also (CD) will be perpendicular to the same Plain. Fig. 14.

It might be taken for granted. If the Demonstration be requir'd, it is as follows.

BD, AD being drawn; in the Plain EF make GD perpendicular to BD. It will also (see the Demonstration of Prop. 6. l. 11.) *be perpendicular to AD. Therefore (*per 4. l. 11.) *GD will be perpendicular to the Plain ABD, that is (by the foregoing* Coroll.) *to the Plain CBDA. Wherefore* (per Def. 3. l. 11.) *CDG is a right Angle. But the Angle CDB is also a right one (forasmuch as with ABD which* (per Defin. 3. l. 11.) *is a right Angle, it maketh two right ones* (per 27. l. 1.) *). Therefore (*per 4. l. 11. *) CD is perpendicular to the Plain GDB or EF.* Q. E. D.]

PROP. IX. Theorem.

RIGHT Lines (AB, EF) which are parallel to the same right Line (CD) although they be not in the same Plain with it, are also parallel betwixt themselves. Fig. 16.

Although it might be taken for granted, yet we will demonstrate it thus.

In the Plain of the Parallels AB, CD, draw GK perpendicular to CD. Likewise in the Plain of the Parallels EF, CD draw HK perpendicular to CD. Therefore (*a*) CK is perpendicular to the Plain GKH. Therefore, seeing AG, EH, be parallel to CK, the same AG, EH (*b*) will be perpendicular to the Plain GKH. Therefore AG, EH (*c*) are parallel. *Q. E. D.*

[a] Per 4. l. 11.
[b] Per 8. l. 11.
[c] Per 6. l. 11.

PROP. X. Theorem.

Fig. 17. *IF two right Lines (AC, BC) be parallel to two right ones (DF, EF); albeit they be not in the same Plain, they comprehend equal Angles (C and F.)*

Let CA, CB, be made equal to FD, FE, and let DE, AB, DA, FC, EB be drawn. Seeing AC, FD are parallel and equal, AD also and CF will (*a*) be parallel and equal. In like manner I might shew BE, CF to be parallel and equal. Therefore AD, BE, are also parallel (*b*) and equal. (*per axiom.* 1.) Therefore (*per* 33. *l.* 1.) AB, DE, are equal. Seeing therefore the Triangles BAC, EDF are equilateral to each other, the Angles C and F (*c*) are equal. *Q. E. D.*

(a) Per 33. *l.* 1.

(b) By the foregoing.

(c) Per. 8. *l.* 1.

PROP. XI. Problem.

Fig. 18. *TO draw a Perpendicular to a given Plain (AB) from a Point given without it (C.)*

The Construction. In the Plain AB draw any right Line as DF, unto which, from C erect the Perpendicular CE. Then in the Plain AB through E draw AEM perpendicular to the same DF. Then to AM from C draw the Perpendicular CG. I say that CG is perpendicular to the Plain.

Through G let HG be drawn parallel to DF. By the Construction DE is perpendicular to CE and EM. Therefore DE is perpendicular to the Plain EM (*d*), as also is HG (*e*). Therefore (by *Defin.* 3. *l.* 11.) CG is perpendicular to HG. But CG by the Construction is also perpendicular to EM. Therefore (*f*) CG is perpendicular to the Plain AB. Which was the Thing propos'd.

[d] Per 4. *l.* 11.

(e) Per 8. *l.* 11.

(f) Per 4. *l.* 11.

Fig. 20. *l.* 11. [Scholium. *In Practice thus. Let there be a Cord or Rule fastned to the given Point A. And from the same, let there be describ'd by the end of it B in the Plain given the Circle BCFL. The Line AK which connects the given Point and the Center of the Circle, will be perpendicular to the given Plain.*

PROP.

PROP. XII. Problem.

FRom a given Point (A) in any Plain (EF) to erect a Line perpendicular to the same Plain. Fig. 19.

From any Point D without the Plain EF make DB (by the foregoing) perpendicular to the Plain EF. And BA being join'd, draw AC parallel to DB. I say the thing is done. The Demonstration is manifest from *Prop.* 8.

Scholium.

IN practice, from the given Point a Perpendicular is erected to the given Plain, if a Square OKN be applied to the given Point [and be turn'd round.]

PROP. XIII. Theorem.

LINES drawn from the same Point cannot be both perpendicular to the same Plain (AB). Fig. 20.

For if they were, they would by 6. be parallel. Which cannot be.

PROP. XIV. Theorem.

IF the same right Line (AB) be perpendicular to two Plains (FG, LQ); the Plains will be parallel. Fig. 21.

Let there be taken in either of the Plains as FG, any Point C, from which let CE be drawn parallel to AB, and meeting the Plain LQ in E. Then CE (*per* 8. *l.* 11.) will be perpendicular to both Plains FG, LQ. Wherefore if AC, BE, be join'd, the Angles A, B, (by *Def.* 3. *l.* 11.) will be right ones. Therefore (*per* 29. *l.* 1.) AC, BE, are parallel. Therefore ACEB is a Parallelogram; and consequently CE, which hath been alrea-

dy shewn to be perpendicular to both Plains, is equal (*per* 34. *l.* 1.) to AB. In the same manner I might shew that all the Perpendiculars to both Plains are equal. Therefore (by *Defin.* 8. *l.* 11.) the Plains are parallel. *Q. E. D.*

PROP. XV. Theorem.

Fig. 22. *IF two right Lines (BA, CA) touching each other be parallel to two other right Lines which also touch one another (ED, FD); the Plains likewise which are drawn through them will be parallel.*

From A let there be drawn AG perpendicular to the Plain EF, and let GH, GI, be parallel to DE, DF. These (*per* 9. *l.* 11.) will also be parallel to AC, AB. Seeing therefore the Angles IGA, HGA, be (by *Defin.* 3. *l.* 11.) right; CAG, BAG, will also (*a*) be right Angles. Therefore GA which is perpendicular to the Plain EF, will also be perpendicular to the Plain BC (*b.*) Therefore the Plains BC, EF, are (by the foregoing) parallel. *Q. E. D.*

(a) Per 27. *l.* 1.

(b) Per 4. *l.* 11.

PROP. XVI. Theorem.

Fig. 23. *Plain (EHFG) cutting parallel Plains (AB, CD) makes the Sections in them (EH, GF) parallel.*

If not, seeing they be in the same intersecting Plain, they will meet somewhere (by *Schol. Prop.* 31. *l.* 1.) as in I. Wherefore seeing the whole Lines HEI, FGI be in the Plains † AB, CD, produc'd, these Plains also will meet in I. Which is absurd, and contrary to *Defin.* 8. *l.* 11.

* Per 1. *l.* 11.

PROP.

PROP. XVII. Theorem.

Parallel Plains cut right Lines (BD and GH) proportionally. Fig. 24.

Let the right Lines BH, GD be drawn in the Plains PQ, TV; and likewise let BG be drawn meeting the Plain RS in F, and let FC, FI be join'd. The Plain of the Triangle BGD cutting parallel Plains, makes the Sections CF, DG, parallel (by the foregoing.) Therefore BC is to CD as BF (e) to FG. Again, the Triangle BHG cutting parallel Plains makes the Sections (by the foregoing) BH, FI parallel. Therefore HI is to IG as (f) BF to FG; that is, (as I have already shew'd) as BC is to CD. *Q. E. D.*

[e] Per 2. l. 6.

[f] Per 2. l. 6.

PROP. XVIII. Theorem.

If a right Line (FE) be perpendicular to a Plain (AB); all the Plains which are drawn through it are perpendicular to the same Plain (AB.) Fig. 25.

Let the Plain GC be drawn through FE making CD the common Section with AB; and let the Lines HK be drawn in the Plain GC, perpendicular to the common Section CD. Now seeing by the Construction HK is perpendicular to the same common Section to which FE is perpendicular by the Hypothesis, KH and FE must be parallel (by 29. *l.* 1.) Therefore HK is also perpendicular to the Plain AB (*per* 8. *l.* 11.) Therefore the Plain GC is perpendicular to the Plain AB (*per Defin.* 4. *l.* 11.)

PROP. XIX. Theorem.

If two Plains (MF, GD) cutting each other be both perpendicular to the same Plain (AB); their Fig. 26.

their common Section also will be perpendicular to that Plain (AB).

For seeing by the Hypothesis the Plain M F is perpendicular to the Plain A B; it is manifest by Definition 4. that there may be drawn in the Plain M F from the Point L a perpendicular to the Plain A B. Again, by the Hypothesis G D is perpendicular to that Plain A B, and therefore there may be also drawn in the Plain G D from the Point L a perpendicular to the Plain A B. But from the Point L (*a*) there can be erected only one Perpendicular to the same Plain A B. Therefore the Perpendicular to the Plain A B, which is drawn from the Point L, must be found in both the Plains M F and G D, and consequently L K, the common Section of those two Plains M F and G D, is perpendicular to the Plain A B. *Q. E. D.*

(a) Per 13. l. 11.

PROP. XX. Theorem.

Fig. 27.

IF a solid Angle (A) is contain'd under three plain Angles (BAC, CAD, DAB); any two of these is greater than the third.

If the three Angles be equal, the Assertion is manifest at first Sight; and it is as certain, if they be unequal. For let B A D be the greatest; and from B A D cut off B A E equal to B A C, and make the Line A C equal to A E. And through E let there be drawn a right Line meeting A B and A D in B and D, and let B C, D C be join'd. Because (by the Construction) the Angles B A E, B A C are equal, as likewise the Sides B A, A E, equal to the Sides B A, A C, the Bases also B E, B C, will be equal (*b*). And because B C, C D (*c*) are greater than B D, the Equals B E, B C, being taken away, there remains C D greater than E D. But the Sides E A, A D, are (by the Construction) equal to the Sides C A, A D. Therefore the Angle (*d*) C A D is greater than the Angle E A D. Seeing therefore the Angle B A C is equal by the Construction to the Angle B A E; those two Angles together B A C, C A D are greater than the whole B A D. *Q. E. D.*

[b] Per 4. l. 1.
(c) Per 20. l. 1.
(d) Per 25. l. 1.

PROP.

PROP. XXI. Theorem.

THE Plain Angles constituting any solid Angle whatsoever are less than four right ones. Fig. 28.

Let A be a solid Angle; let the right Lines BC, CD, DE, EF, FB, be subtended to the plain Angles which make up the solid one, so that those right Lines be all in one Plain. Which being done, there is constituted a Pyramid, whose Base is the Polygon BCDEF; A is the Top, and it is contain'd under so many Triangles G, H, I, K, L, as there are plain Angles which compose the solid Angle A. And now because the two Angles ABF, ABC, are (by the foregoing) greater than the third FBC; and the two ACB, ACD, are greater than the third BCD, and so on: All the Angles of the Triangles G, H, I, K, L, about the Base, as taken together, are greater than all the Angles of the Base B, C, D, E, F, taken together. But the Angles of the Base together with four right ones, make twice so many right Angles (by *Theorem* 2. *Schol.* after 32. *l.* 1.) as there are Sides, or, which is the same, as there are Triangles. Therefore all the Angles of the Triangles about the Base, together with four right ones, make more than twice so many right Angles, as there are Triangles. But the same Angles about the Base, together with the Angles that compose the Solid, make up (*a*) twice so many right Angles as are the Triangles. It is manifest therefore that the Angles which compose the solid Angle A are less than four right ones. *Q. E. D.*

(a) *Per* 32. *l.* 1.

Corollary.

FRom this and the foregoing it is obvious to collect, that a solid Angle may be compos'd of any three plain Angles, which are less than four right ones, if so be that any two of them be greater than the other.

Scho-

Scholium.

FRom this Proposition is demonstrated that famous Theorem, That only three regular and equal plain Figures can contain a Body; to wit, equilateral Triangles, either 4, or 8, or 20. 6 Squares, and 12 Pentagons. And consequently that there are only five regular Bodies. A Pyramid which is contain'd under 4; an Octaedrum which is comprehended by 8. And an Icosiedrum, which is bounded by 20 equilateral Triangles. A Cube which is contain'd under 6 Squares; and the Dodecaedrum under 12 regular and equal Pentagons. Now a Body is called Regular which is comprehended under regular and equal Plains.

Demonst. A solid Angle cannot be compos'd of only two equilateral Triangles; three at least are requir'd.

Of three equilateral Triangles meeting in one Point, there may be constituted the solid Angle of a Pyramid; of four, the solid Angle of an Octaedrum; of five, the solid Angle of an Icosiedrum: Forasmuch as both 3, 4, and 5 Angles of an equilateral Triangle, are less than 4 right ones, as is gathered from *Coroll.* 12. *prop.* 32. *l.* 1.

And because three Angles of a regular Pentagon (as is gathered from *Coroll. prop.* 11. *l.* 4.) are less than four right ones, three Pentagons meeting in one Point will constitute a solid Angle; that of the Dodecaedrum.

That of three Squares meeting in one Point, may be compos'd the solid Angle of a Cube, is manifest of it self. And thus there arise five regular Bodies.

But that there are no more than these five is thus proved.

Six Angles of an equilateral Triangle make just four right ones. For one is two Thirds of one right one; and therefore six such will make (by *Coroll.* 12. *prop.* 32. *l.* 1.) twelve Thirds of one right one, that is, four right ones. And therefore of six equilateral Triangles a solid Angle cannot be compos'd; much less of more.

That of four Squares a solid Angle cannot be made, much less of more, is manifest in it self.

Four Angles of a regular Pentagon are greater than 4 right ones. For (by *Coroll. prop.* 11. *l.* 4.) each of them makes six Fifths of one right one. Therefore a solid Angle can-

cannot be made of four such Pentagons; much less of more.

Nor can a solid Angle be compos'd of any other regular Figures whatsoever. Three Angles of a regular Hexagon (by *Coroll.* 2. *prop.* 15. *l.* 4.) are equal to four right ones. For one makes four Thirds of one right one; and therefore three make twelve Thirds of one, that is, four entire right ones. Therefore of three Hexagons a solid Angle cannot be made up; much less of more.

But seeing three Angles of a regular Hexagon are equal to four right ones, three Angles of any other Figures whatever greater than an Hexagon, as of an Heptagon, Octagon, *&c.* will be greater than four right ones. Wherefore it is manifest that the rest of the regular Figures are all incapable of composing a solid Angle; and consequently that there can be no regular Bodies besides the five foregoing.

PROP. XXII, XXIII.

ARE very prolix, and tedious to Beginners, and scarce at any time come into Use.

PROP. XXIV. Theorem.

THE Plains which contain a Parallelepiped are (1.) Parallelograms. (2.) The opposite ones are equal. (3.) The Plains are equal. Fig. 22.

Part I. The Plain AF cutting the Plains BD, FH, which by *Defin.* 13. are parallel, makes (*a*) the Sections BA, FE, parallel. Again, the Plain AF cutting the Plains AH, BG, which by the same Definition are parallel, (by the same) makes the Sections AE, BF, parallel. Therefore BAEF is a Parallelogram. By the like Argument the rest of the Plains of the Parallelepiped may be prov'd to be Parallelograms. (a) *Per* 16. *l.* 11.

Part II. Because it is manifest from the first Part, that AB, BC, are parallel to EF, FG, the Angles ABC, EFG, must be (*b*) equal. Wherefore seeing the (b) *Per* 10. *l.* 11.

the alternate Sides also are equal, the opposite Parallelograms BD, FH, are like or similar. And the same of the rest.

Part III. This is manifest from the first Part, and 4, or 8th of the First Book.

PROP. XXV. Theorem.

Fig. 30.

IF a Parallelopiped (G F D I) or any Prism whatever be cut by a Plain (NP) that is parallel to the opposite Sides; there will be this Proportion, as the Base (DCPO) is to the Base (OPFE) so is the solid (GP) to the solid (NF.)

This is demonstrated in the same manner as *p.* 1. *l.* 6.

Coroll.

A Prism cut by a plain Parallel to the opposite Plains, hath a Section like and equal to the opposite Plains.

PROP. XXVI, XXVII.

ARE *not necessary.*

PROP. XXVIII. Theorem.

Fig. 29.

A Plain passing through the Diameters of opposite Plains (AC, EG) cuts the Parallelopiped into two equal Prisms.

[a] Per 24. l. 11.
(b) Per 9. l. 11.
(c) Per 7. l. 11.

Because (*a*) BG, BE, are Parallelograms; CG, AE, are equi-distant from the same BF. Therefore (*b*) they are also parallel betwixt themselves, and consequently are in one Plain. Therefore the right Lines AC, EG, are (*c*) in one Plain. But now that a Plain drawn thro' them doth cut the Parallelopiped into two equal Prisms,

is thus shew'd. Let the Prism AEGCDH be understood to be so constituted upon its Plain EACG, that the Angles D, H, bend towards the Angles B, F. It is manifest that it will yet be betwixt the parallel Plains BADC, FEHG. But then D must needs fall upon B, and H upon F. For, let D fall without B, if it may be, and in N. The Angle BAC (*d*) is equal to the Angle DCA. But DCA is equal to NAC (for it is one and the same Angle). Therefore BAC and NAC are equal: Which is absurd. Therefore D falls upon B; and for the same Cause H upon F. Therefore the Prism AEGCDH coincides with the Prism ACGEFB, and consequently they are equal (by *Axiom.* 7.)

(d) Per 27. *l.* 1.

PROP. XXIX, XXX. Theorems.

THE Parallelopipeds (FEAGKIMC) and FEBHLOMI) which have the same Base (EFIM) and the same Altitude, and consequently exist between parallel Plains (EFIM) and (GAOL) are equal. Fig. 31.

For they either exist betwixt the lateral parallel Plains EAOM and FGLI, or not. Let the first be suppos'd. From the 24th of this, and the 8th of the first Book, It is manifest that the Triangles AEB, CMO; likewise GFH, KIL, are equilateral and equiangular to each other. Wherefore, as in the foregoing, I might shew that the Prisms CMOLIK, AEBHFG, being laid upon each other will coincide, and consequently are equal. Wherefore the common solid FEBHKCMI being added, the whole Parallelopipeds FEAGKIMC and FEBHLOMI are equal. *Q. E. D.*

Then let the Parallelopiped FXQEMIPR not exist betwixt the same lateral parallel Plains with the Parallelopiped FEAGKCMI. Here, because by the Hypothesis, GK, AC, RP, QX, are in one Plain, which is parallel to the Base EFIM; let RP, QX, cut GK in L and H, and AC in O and B; and let EB, MO, FH, IL, be join'd. It is easy now to shew that the Plains containing the Solid FEBHLOMI are Parallelograms, the opposite ones of which are equi-distant, and

and consequently that that Solid is (by *Defin.* 13. *l.* 11.) a Parallelopiped. But to this by the first Part the Parallelepipeds FXQEMIPR, and FEAGKCMI, are each of them equal. Therefore they are also equal betwixt themselves. *Q. E. D.*

Corollary.

THis Proposition is like to the 35th of the first Book; for it affirms concerning Solids, what that doth touching Plains. Wherefore the Demonstration of the rest of the Cases will be like also.

PROP. XXXI. Theorem.

Fig. 33. *PArallelopipeds upon equal Bases (AO and EG) and in the same Altitude (S) are equal.*

First, let the Parallelopipeds have their Sides perpendicular to the Bases. Unto the side FG produc'd let there be made a Parallelogram GMKH equal and like to the Parallelogram AO; and the Parallelogram GMPR being perfected, let the right Lines PM, RG meet KH in Q and L. And now let Parallelopipeds be understood to be constituted upon GK, GQ, GP, whose Sides are perpendicular to the Bases, and S is their common Altitude. The Solid EGS is to the Solid GPS, as EG (*per.* 25. *l.* 11.) is to GP; that is, (because EG, AO, are equal by the Hypothesis), as AO to GP; that is, by the Construction, as GK is to GP; that is, as GQ is to GP (*per* 35. *l.* 1.); that is, as the Solid GQS is to the same Solid GPS. (*per* 25. *l.* 11.) Because therefore the Solids EGS and GQS have the same Proportion to the Solid GPS, the Solid EGS will be equal to the Solid GQS; that is, to the Solid GKS (*per* 29. *l.* 11.) that is, (because the Bases GK, AO, are equal and like by the Construction) to the Solid AOS, as it appears from 29. *l.* 11. and even in it self. Which was the thing propos'd. Note, that in this reasoning, he Solids are suppos'd to be right or perpendicular ones.

Then let the given Parallelopipeds EGS, AOS have their Sides at the Bases EG, AO, oblique. Let there now be made upon EG, AO, Parallelopipeds, whose Sides are perpendicular to the Bases in the heighth S; these will be equal to the oblique ones by 29th or 30th. Wherefore seeing by the first Part right Parallelopipeds are equal betwixt themselves; the oblique ones will be equal betwixt themselves likewise. Q. E. D.

PROP. XXXII. Theorem.

ALL Parallelopipeds whatever of equal Heighth, are betwixt themselves as their Bases. Fig. 34.

Let GO and A be the Bases. Upon CO make the Parallelogram OE equal to A.

Upon BC, OE, let Parallelopipeds be understood to be erected in the Altitude K; these therefore will be Parts of one Parallelopiped BEK. Therefore the Parallelopiped OEK, is to the Parallelopiped BCK, as the Base OE, to the Base BC (*per* 25. *l.* 11.); that is by the Construction, as the Base A is to the Base BC. But because the Bases OE and A, are equal, the Parallelopipeds OEK and AK are equal (by the foregoing). Therefore also the Parallelopiped AK is to the Parallelopiped BCK, as the Base A is to the Base BC. *Q. E. D.*

Scholium.

THAT which hath here been shew'd of Parallelopipeds, will be demonstrated in the Twelfth Book of Pyramids, *Prop.* 6; Of all Prisms whatever, in *Coroll.* 1. after *Prop.* 9; Of Cones and Cylinders, *Prop.* 11.

PROP. XXXIII. Theorem.

LIKE Parallelopipeds (HA and CM) are in a triplicate Proportion of their homologous Sides (AB, BC). Fig. 35.

Let the Parallelepipeds AH, CM, be like. Therefore all their Plains (by *Defin.* 9. *l.* 11.) are like; and Consequently AB (by *Defin.* 1. *l.* 6.) is to BC, as EB to BO; and as FB is to BG, so is EB to BO. Moreover the Angles of the Plains are also equal (by the same). Therefore let the Solids AH, CM, be so plac'd, that the equal Angles CBO, ABE, may be opposite, and the Sides AB, CB, may lye so as to make one straight Line; and then EB, OB will also lye so as to make one straight Line. Now let Solids be imagin'd to be constituted upon the Plains BQ and EC, in such sort that the Solids KB, HA, may be one Parallelopiped, and KB, PO, may make one Parallelopiped, and PO, CM, may make one Parallelopiped likewise. The Solid HA is to the Solid KB (*per* 25. *l.* 11.) as AE to BR; that is, (*per* 1. *l.* 6.) as AB to BC; that is, (as I shew'd above by the Hypothesis,) as EB is to BO; that is, (by the same) as EC is to BQ; that is, (*per* 25. *l.* 11.) as the same Solid KB is to the Solid PO. Therefore the three Solids HA, KB, PO, continue the same Proportion. But now the Solid KB is to the Solid PO (by the same) as the Base BR is to the Base PQ; that is, (*per* 1. *l.* 6.) as EB is to BO; that is, as FB is to BG, as it was shew'd above by the Hypothesis; that is, (by the same) as the Plain FC is to the Plain BS; that is, (*per* 25. *l.* 11.) as the same Solid PO again is to the Solid CM. Therefore the four Solids, HA, KB, PO, CM, are continually proportional. Therefore (by *Defin.* 10. *l.* 5.) the Proportion of the first HA to the fourth CM is triplicate of the Proportion of the first HA to the second KB; that is, triplicate to the Proportion (*per* 25. *l.* 11.) of AE to BR; that is, triplicate (*per* 1. *l.* 6.) to the Proportion of the homologous Sides, AB to BC. *Q. E. D.*

[Coroll. (1.) *Hence if there be four right Lines continually proportional; as is the first to the fourth, so is a Parallelepiped describ'd upon the first, to a Parallelepiped like, and in like manner describ'd upon the second.*

(2.) *Upon this also depends that most famous Problem concerning doubling the Cube; of which afterwards. Schol. p.* 18. *l.* 12.

(3.) *Hence also is to be corrected the Error of those, who suppose that the Proportion of like Solids is the same as is that of their Sides. For the Cube of a Line which is double to another Line, is not only double to the other, but as* 8 *to one.*

And

And the Cube of a Line, which is treble to another Line, is not only treble to the other Cube, but contains it 27 Times. For 1 : 2 : 4 : 8 ÷ and 1 : 3 : 9 : 27 ÷, and the same Thing is to be said of all like Bodies whatsoever; as will appear afterwards.

(4.) *Hence the triplicate Proportion of any Quantities whatsoever is the Proportion of the Cubes of the same Quantities. Let there be any two Quantities in the triplicate Proportion of the Quantities AB, BC; they shall also be as AB Cube is to BC Cube.*]

Scholium.

THAT which hath here been shew'd of Parallelepipeds will be demonstrated Book 12. Of Pyramids *Prop.* 8. Of all Prisms whatsoever, *Coroll.* 2. *Prop.* 9. Of Cones and Cylinders, *Prop.* 12. Of Spheres, *Prop.* 18.

PROP. XXXIV. Theorem.

Fig. 36.

IF the Parallelepipeds (BM, CK) be equal, their Bases and Altitudes are reciprocally proportional; (that is, the Base AM is to the Base FK, as reciprocally the Height FC is to the Height AB). And if they be reciprocally proportional their Bases and Altitudes are equal.

Part I. First let the Sides be perpendicular to the Bases. If now the Altitudes of the Solids BM, CK be equal, the thing is manifest.

If the Altitudes be unequal, from the greater FC cut off FE equal to BA: and through E draw the Plain EL parallel to FK. The Base AM is to the Base FK, (*per* 25. *l.* 11.) as the Solid BM is to the Solid EK; that is, (because by the Hypothesis the Solids BM, CK are equal) as the Solid CK is to the Solid EK; that is, (by the same) as CG is to EG; that is, (*per* 1. *l.* 6.) as CF is to EF; that is, by the Construction, as CF to BA. *Q. E. D.*

Then let the Sides be oblique to the Bases. Let right Parallelopipeds be erected upon the same Bases in the

ſame Heighth. The oblique Parallelopideds will be equal to theſe. Wherefore ſeeing theſe by the firſt Part have their Baſes and Altitudes reciprocal, thoſe alſo will be likewiſe. *Q. E. D.*

Then let the Sides be oblique to the Baſes. Let right Parallelopipeds be erected upon the ſame Baſes in the ſame Heighth. The oblique Parellelopipeds will (*per.* 29. *and* 30. *l.* 11.) be equal to theſe: wherefore ſeeing theſe by the firſt Part have their Baſes and Altitudes reciprocal thoſe alſo ſhall be ſo likewiſe. *Q. E. D.*

Part II. Let the Altitudes be unequal, and the Sides perpendicular to the Baſes; And from the greater CF take EF equal to AB. The Solid BM is to the Solid EK, (*per* 32. *l* 11) as AM is to FK; that is by the Hypotheſis as CF is to AB; that is, by the Conſtruction, as CF is to EF; that is, as CG is to (*f*) EG; that is, (*g*) as the Solid CK is to the ſame Solid EK Therefore the Solids BM and CK have the ſame Proportion to EK. Therefore they are equal. *Q. E. D.*

(f) *Per* 1. *l.* 6.
[g] *Per* 25. *l.* 11.

Corollaries.

WHAT Affections have been demonſtrated of Parallelopipeds, *Prop.* 29, 30, 31, 32, 33, 34, do alſo agree to triangular Priſms, which are the Halves of Parallelopipeds. As is manifeſted from *Prop.* 28. Therefore,

Fig. 37. 1. The triangular Priſms, which are of equal Heighth, are as their Baſes, A, B.

2. If they be like, their Proportion is triplicate to the Proportion of the Sides, oppoſite to equal Angles.

3. If they be equal, they reciprocate their Baſes and Altitudes; and if they reciprocate their Baſes and Altitudes they are equal.

Scholium.

WHAT hath here in *Prop.* 34. been ſhew'd of Parallelopipeds, will be demonſtrated in the 12th Book of Pyramids, *Prop.* 9; Of all Priſms whatſoever, *Coroll.* 3. after *Prop.* 9; Of Cones and Cylinders, *Prop.* 15.

PROP.

PROP. XXXV.

IS very long, and subservient to the following Proposition, which we will demonstrate without it.

PROP. XXXVI. Theorem.

A Parallelopiped (DH) made of three proportional Right Lines (A, B, C,) is equal to a Parallelopiped (IN), which is made of the Mean (B), and is equiangular to the former. Fig. 38.

Let the Base FD of the Parallelopiped DH have the Side EF equal to A, and the other Side ED equal to C: And the Side EG which stands upon the Base equal to B. Thus the Parallelopiped DH will be made of the hree right Lines, A, B, C. Then let the Three Sides, LX, IX, XM, (and consequently all the rest) of the Parallelopiped IN be equal to the middle Line B: And the solid Angle X equal to the solid Angle E; the Parallelopiped IN will be made of the Mean B, and be equiangular to the former. I say also that it is equal.

For seeing by the Hypothesis and the Construction as FE is to LX, so reciprocally IX is to DE, the Bases also * DF, IL will be equal. Now because the solid Angles at E and X are equal; if they be put within one another, † they will coincide; and because of the Equality of the right Lines, EG, XM, the Points M and G, will coincide. Wherefore both the Solids will have one perpendicular Altitude; to wit, the right Line which is let fall from the Points M, G, (now become one) unto the Plain of the Base. The Solids therefore DH, IN * are equal. Q. E. D.

* Per 14. l. 6.
† Per defin. 11. l. 11.
* Per 32. l. 11.

Scholium.

WE will further observe what is of great Use, that of three Lines drawn into or multiplied one by another

ter what manner soever, a Solid of the same Magnitude is produc'd.

ABC. CAB. BCA.
1. 2. 3.

In the present Scheme the two first Letters design the Base; the third the Altitude. Let us compare the first with the second.

The Base A B is to the Base C A, *per* 1. *l.* 6. as the Side B is to the Side C; that is, reciprocally as the Heighth B is to the Heighth C. Therefore by *p.* 34,

A B C, is equal to C A B.

In the same manner it may be shew'd that the first is equal to the third, and the third to the second.

PROP. XXXVII. Theorem.

Parallelopepids which are like, and describ'd in the like manner by proportional right Lines, will themselves also be proportional; and conversely.

This is manifest of it self. For the Proportions of Parallelopepids by the 33d of this Book will be triplicate to those Proportions which by the Hypothesis are equal which the Lines have betwixt themselves.

The Converse is manifest of it self also.

The Proposition is true of all sorts of like Bodies, which will appear from Book the 12th to have betwixt themselves a Proportion triplicate to that which the Sides have.

PROP. XXXVIII, XXXIX.

THESE contain nothing remarkable, and are scarce of any Use.

PROP. XL.

THIS is of small Use, and indeed no other than the 28th Proposition in another View.

Sho-

Scholium.

FRom what hath hitherto been demonstrated we have the Dimension of triangular Prisms, and of quadrangularor parallelopepids; to wit, if the Altitude be multiplied into the Base. As if the Altitude be of 10 Feet, and the Base of a 100 square Feet (now the Base is measured by *Schol. p.* 36, or 41. *l.* 1.) multiply 10 by 100 there will arise 1000 cubick Feet for the Solidity of the given Prism.

The Demonstration is easy. For like as a Rectangle ariseth from the Multiplication of one Side by another, so a right Parallelopiped is produc'd from the Heighth drawn into the Base. Therefore every Parallelopiped is also produc'd fr m the Altitude multiply'd into the Base, seeing by 31. *l.* 11. it is equal to a right Parallelopiped, constituted upon the same Base with the same Heighth.

Then seeing the whole Parallelopiped is produc'd from the Heighth into the whole Base; the half of the Parallelopiped (that is, a Triangular Prism by 28. *l.* 11.) will be produc'd from the Altitude multiplied by half the Base; to wit, the Triangle ILK.

The Elements of *Euclid.*

BOOK XII.

With Us the EIGHTH.

WHAT in the foregoing Books we have endeavoured to perform; namely, To bring the Elements of the Mathematicks into a more easy and brief Method; will be to be endeavour'd in this Twelfth Book especially; the Doctrine whereof is most necessary, but the Demonstrations are so prolix, that they commonly make Beginners almost to despair. We have so propos'd to our selves to remedy this Evil, that in the mean while we will not depart from the Rigor of Geometrical Demonstration. Which Thing whether or no we have attain'd, the Reader will understand, if he shall compare this of ours with *Euclid*'s Prolixity.

NOW after Euclid *had in the former Book declared the Elements of Solids, and defined the Measures of the most easy Bodies, those, namely, which are terminated with plain Surfaces; In this Twelfth Book he considers Bodies bounded with curve Surfaces; to wit, Cylinders, Cones, and Spheres; compares them betwixt themselves; and defines their Measures. This Book is indeed most profitable, because it contains those Principles, on which the chief Masters of Geometry, and especially* Archimedes, *have built so many Famous Demonstrations, concerning the Cylinder, Cone, and Sphere.*

DE-

DEFINITIONS.

1. A Pyramid is a Solid (Z L) comprehended under the Triangles (A L C, C L F, F L B, B L A) plac'd from one Plain (Z) to one Point (L). *Fig.* 1. *l.* 12.

The Plain Z is call'd the Base, and may be either a Triangle or Quadrangle, or any other Figure; from each of the Sides whereof there arise Triangles meeting together in the Point L, which is call'd the Vertex or Top.

As the Triangle amongst rectilineal plain Figures, so the Pyramid amongst solid ones is the first and most simple.

2. If without the Plain of some Circle (C L) there shall be taken the Point (A), and from it be drawn the infinite right Line (A F) touching the Circle in C; and this Line, the Point (A) remaining fix'd, be turn'd about the Circumference of the Circle, until it returns thither from whence it began to be moved; the Surface describ'd by the right Line (A C F) is term'd a conical Surface, and the Body which is contain'd under this Surface, and the Circle (C L) is call'd a Cone. *Fig.* 2, 3.

The Vertex of the Cone is (A).

The Circle (C L) is the Base of the Cone.

The right Line (A B) drawn from the Vertex to the Center of the Base is the Axis of the Cone.

The Side of the Cone is the right Line (A C) drawn from the Vertex to the Circumference of the Base, which that it is wholly in the Surface of the Cone is manifest from the Production of the Figure.

A right * Cone is when the Axis (AB) is perpendicular to the Base. * *Fig.* 3.

A scalene † or oblique Cone, is when the Axis (A B) is not perpendicular to the Base. † *Fig.* 4.

A right Cone is also made by a right-angled Triangle (C B A) turn'd round about one of the perpendicular Sides (A B). See *Fig.* 2.

3. If an infinite right Line (C O F) be turn'd about two Circles (C L, O Q) equal and parallel, untill it returns to that Place from whence it began to be mov'd, and remains always, whilst it is mov'd, parallel to it self, the Surface describ'd by the right Line (C O F) is called *Fig.* 4, 5.

a Cy-

a Cylindrical Surface; and the Body which is contain'd under this Surface, and the two Circles is call'd a Cylinder.

The Bases of the Cylinder are the Circles (C L, O Q) the right Line (A B) which connects the Centers of the Bases is called the Axis. The right Line (O C) in the Surface of the Cylinder, touching both the Bases, is called a Side of the Cylinder.

Fig. 4. A right Cylinder is when the Axis is perpendicular to the Base.

Fig. 5. A Scalene or oblique Cylinder, is when the Axis is not perpendicular to the Base.

A right Cylinder is also made by a Rectangle (O C B A) turn'd round about one Side (B A). See *Fig.* 4.

Fig. 20, 21. 4. Like Cones and Cylinders are those, which have their Axes (A K, Z O) and the Bases of their Diameters (B F, Q R) proportional.

5. A Sphere is a Solid contain'd under one Surface, unto which Surface all the right Lines that are drawn from a certain Point within the Figure, are equal amongst themselves. That Point is call'd the Center. The Diameter of the Sphere is a right Line drawn through the Center unto the Surface on both Sides.

Fig. 6. A Sphere is produced if a Semicircle be turn'd about its Diameter (A F) which remains in the mean while unmov'd.

6. Magnitudes inscrib'd into or describ'd about some Figure, whether they be greater or lesser than the Figure, are then said to *end* in the Figure, when they will at the last differ from it by a Quantity less than any given one whatsoever, or how small soever.

Therefore if those Magnitudes which are inscrib'd into some Figure will at last fall short of it by a Deficiency less than any given one whatsoever, the Magnitudes inscrib'd are said to end in the Figure; and if those which are circumscrib'd about some Figure, will at last exceed it by an excess less than any given one whatsoever, they shall be said to End in the Figure.

PROP.

PROPOSITION I. Theorem.

THE Proportion of like Polygons inscrib'd in a Circle, is duplicate to the Proportion of the Diameters (AF, IC). Fig. 6, 7.

Let AO, BF; IR, LC, be drawn. Because the Polygons are suppos'd to be like, the Angles (OBA, RLI) will (*per Defin.* 1. *l.* 6.) be equal; and the Sides OB, BA, proportional to the Sides RL, LI. Therefore in the Triangles OAB, RIL (*per* 6. *l.* 6.) the Angles O and R are equal. Therefore also BFA and LCI which stand upon the same Arches BA, LI, are (*p.* 21. *l.* 3.) equal. But the Angles FBA, CLI, in Semicircles are (*p.* 31. *l.* 3.) right ones. Therefore the other Angles (*p. Coroll.* 9. *pr.* 32. *l.* 1.) BAF, LIC, are equal. Therefore because the Triangles FAB, CIL, are equiangular to each other, they are (*p.* 4. *l.* 6.) like; and BA will be to LI, as AF to IC. Now because by the Hypothesis the Polygons are like, their Proportion will be duplicate (*p.* 20. *l.* 6.) to the Proportion of the Sides BA, LI; that is, as I have already shew'd duplicate to the Proportion of the Diameters AF, IC. *Q. E. D.*

Corollary.

THE Circumferences of like Polygons inscribed in a Circle are betwixt themselves as the Diameters. Fig. 6, 7.

Seeing it hath already been shew'd, that AB is to LI, as AF is to IC; OB will also be to RL, as AF to IC: And so of the rest of the Sides. Therefore all the Sides together will be to all the Sides together; that is, one circumference to another, as AF is to IC.

A Lemma.

POlygons inscrib'd in a Circle end in a Circle. Inscribe a Square, as ACBD. Seeing this is half Fig. 8.

(per

(*per Schol. p.* 6, and 7. *l.* 4.) of the Square which is circumscrib'd, it will be greater than half of the Circle. Wherefore if this be taken out of the Circle, there will be taken out of it more than half. Then each Arch being bisected in E, K, I, H, inscribe an Octagon: And let FG touch the Circle in E, which FG let BC, DA meet in G and F; CF will be a Parallelogram, of which seeing the Triangle CEA (*per* 41. *l.* 1.) is half, this will be more than half of the Segment CEA. In the same manner each of the Triangles AKD, DIB, *&c.* is more than half each of the Segments. Therefore all the Triangles are more than half all the Segments. Therefore if you take these out of those, that is, out of the Remainder of the Circle, more than half will be taken away. In the same way of arguing if there be inscrib'd in the Circle Polygons of Sides always double in Number; I can shew that there will always be taken out of the Remainder of the Circle more than half. Therefore the Remainder must at last be less than any given one whatsoever; and consequently the inscrib'd Polygons will at last fall short of a Circle by a Quantity less than any given one whatsoever; that is, (*per Defin.* 6. *l.* 12.) will end in a Circle.

PROP. II. Theorem.

Fig. 6, 7. *THE Proportion of Circles is duplicate to the Proportion of their Diameters.*

The Proportion of Polygons inscrib'd in a Circle without End is (*per* 1. *l.* 12.) duplicate to the Proportion of the Diameters. But Polygons (by the foregoing *Lemma*) inscrib'd in a Circle infinitely, at last end in the Circle. Therefore the Proportion of Circles is also duplicate to the Proportion of the Diameters.

PROP. III, IV.

ARE Prolix, and hard for young Beginners, and have no other Use, than that they serve

to

to the Demonstration of the Fifth, which we shall demonstrate much more easily without them.

Lemmata, *or preparatory Propositions to* Prop. V.

Lemma I.

IF two triangular Pyramids be cut with Plains (OSE, RXZ) parallel to the Bases (ABC, IQV), which same Plains divide the Sides (CF, QL) proportionally in (E and Z) (OSE, RXZ) will be betwixt themselves as the Bases (ACB, IQV). *Fig. 9.*

Because the parallel Plains OSE, ABC, are cut by the Plains BFC, AFB, AFC, the common Sections SE, BC, and OS, AB, and OE, AC, will be (*per* 16. *l.* 11.) parallel. Wherefore the Angles OSE, ABC, and SOE, BAC, and OES, ACB, two and two, are (*per* 10. *l.* 11.) equal. Wherefore the Sections OSE, ABC, are like (*per* 4. *l.* 6.) In the same manner I might shew that the Sections RXZ, IVQ, are like. Therefore (*per* 19. *l.* 6.) the Proportion of the Section ABC, to the Section OSE is duplicate to the Proportion of the Side BC, to the Side SE; and the Proportion of the Section IVQ to RXZ is duplicate to the Proportion of VQ to XZ. But the Proportions of BC to SE, and of VQ to XZ are the same (for BC is to SE (by *Coroll.* 1. *per* 4. *l.* 6.) as CF to EF; that is, by the Hypothesis, as QL to ZL; that i,, (by the same *Coroll.*) as VQ to XZ). Therefore the Proportion of ABC to OSE is the same with the Proportion of IVQ to RXZ. *Q. E. P.*

Lemma II.

PRisms inscrib'd infinitely in a Pyramid (ZCAF) which hath a triangular Base, end in the same Pyramid. *Fig. 10.*

Let the Side of the Pyramid be divided into a certain Number of equal Parts AB, BG, GF, and through B and G there being made the Sections GDN and BEP parallel to the Base ZAC; let the triangular Prisms BEPMAO and GDNKBQ be understood to be inscrib'd

scrib'd in the Pyramid. These then being continued without the Pyramid, let there be understood to be describ'd about the Pyramid the Prisms CIBA, PXGB, NHFG. The Excesses of the circumscrib'd Prisms above the inscribed ones are the Solids IM, XK, HG, which taken together are equal to the Prism CIBA: For HG (*per* 25. *l.* 11.) is equal to DB; and consequently HG with XK are equal to PXGB, that is, (by the same) to MEBA. Therefore the three HG, XK, IM, are equal to the whole CIBA. But if AF be divided without End into more equal Parts, and consequently the Number of Prisms be infinitely increas'd, AB will become less than any given Line. Therefore (as is manifest from *p.* 25. *l.* 11.) the Prism CIBA will become less than any given one. Therefore the excess of the circumscrib'd Prisms, (and much more of the Pyramid ZCAF which is part of the Prisms circumscrib'd about it) above the inscribed Prisms will be less than any given Prism. Therefore the inscrib'd Prisms (by *Defin.* 6. *l.* 12.) end at last in a Pyramid. *Q. E. D.*

PROP. V. Theorem.

Fig. 11. *Triangular Pyramids of the same Heighth have that Proportion betwixt themselves, which their Bases (AQR, ESX) have.*

Let the equal Altitudes of the Pyramids be represented by the Sides AP, EZ; which on both Sides let be divided into as many equal Parts as you will, but so that they be of the same Number; and let there be made through the Points of the Divisions Sections parallel to the Bases; let triangular Prisms, of the same Number and the same Heighth be understood to be inscrib'd in both Pyramids. And now because the Prisms, LA, IE, are of the same Heighth, the Prism LA will be to the Prism IE (by *Coroll.* 1. *p.* 34. *l.* 11.) as the Base LOB is to the Base INK; that is, (by *Lemma* 1.) as the Base QRA is to the Base SXE. In the same manner I might shew that each of the Prisms inscrib'd into the Pyramid QPAR, is to each inscrib'd into the Pyramid SZ EX,

EX as the Base QAR is to the Base SEX. Therefore all of them together are to all of them together as Base is to Base. Wherefore seeing they at last end (*per Lem.* 2.) in the Pyramids themselves, the Pyramids themselves also will be as their Bases. *Q. E. D.*

PROP. VI. Theorem.

ALL Pyramids whatsoever which are of equal Heighth have that Proportion betwixt themselves which their Bases (AB, CFO) have. Fig. 12, 13.

Let their Bases be resolv'd into Triangles A, B, C, F, O; and the whole Pyramids into triangular Pyramids. The Pyramid AX is to the Pyramid OZ (by the foregoing) as A is to O; and the Pyramid BX is to the Pyramid OZ as B is to O (by the same). Therefore the Pyramids AX, BX, together (that is, the whole Pyramid ABX) are to the Pyramid OZ as A, B, together are to O. By the same Argumentation the Pyramid ABX is to the Pyramid FZ (by the foregoing) as A, B, are to F: And ABX is to CZ as A, B, is to C. Therefore ABX is to the three OZ, FZ, CZ together; that is, to the whole Pyramid OFCZ, as A, B, together is to O, F, C, together. *Q. E. D.*

PROP. VII. Theorem.

EVery Pyramid is the third Part of a Prism which hath the same Base and Heighth.

First, let the triangular Pyramid BGAC have the same Base and Heighth with the Prism BACFEO: Let BF, AO, AF, be drawn. The Triangles BFC, BFO are (*per* 34. *l.* 1.) equal. Therefore the Pyramid BFCA, is equal to the Pyramid BOFA. For the same Reason OEAF is equal to the Pyramid OBAF; that is, to the Pyramid BOFA, for they are the same Pyramids. Therefore BFCA, and OEAF, are also equal. Fig. 14.

equal. Therefore all three BFCA, OEAF, OBAF, or BOFA, are equal. Therefore the three together are triple of one BFCA. But those three constitute the Prism BACFEO. That Prism therefore is treble to the Pyramid BFCA; that is, (*per* 5. *l.* 11.) to BGAC. *Q. E. D.*

Fig. 15. Then let any Pyramid whatsoever have the same Base and Heighth with the Prism AEFH: the Lines BC, BO, BE, and NI, NG, NH, being drawn, resolve the Prisms into triangular Prisms, and the Pyramid into triangular Pyramids. Which being done the Demonstration is manifest from the first Part. For each Part of the Prisms will be treble of each Part of the Pyramids. And consequently the whole Prism will be treble to the whole Pyramid. *Q. E. D.*

PROP. VIII. Theorem.

Fig. 16. *THE Proportion of like Pyramids (OACB, KHIN) is triplicate to that which the homologous Sides (AB, HN,) have to each other.*

First, let them be triangular: The Parallelograms AM and HQ being perfected, set upon them the Parallelopipeds AG, HL, in the Heighth of the Pyramids; which, seeing the Pyramids are like, will also (as appears from *Defin.* 9. *l.* 11.) be like. Then let EF, RP, be drawn; and through EF, CB, as likewise through RP, IN, the Parallelopipeds will be cut (*per* 28. *l.* 11.) into two equal Prisms; each of which will be treble to the Pyramids OACB, and KHIN (by the foregoing.) Therefore both together, that is, the whole Parallelopipeds AG, HL will be Sixfold of the Pyramids. Therefore the Pyramids are proportional to the Parallelopipeds. But (*per* 33. *l.* 11.) the Proportion of these each to other is triplicate to the Proportion of the Sides AB, HN. Therefore so likewise is the Proportion of the Pyramids.

Fig. 17. But if the like Pyramids shall be polygonal, let them be resolv'd into the triangular ones AR, BR, CR, and OK, EK, FK. You may from 20. and 5. *l.* 6. and *Defin.*

fin. 9. *l.* 11. easily shew that AR is like to OK, and BR to EK, and CR to FK. Therefore by the former Part, the Proportion of the Pyramids AR, OK, is triplicate to the Proportion of IM to PZ: And the Proportion of the Pyramids BR and EK is triplicate to the Proportion of MX to SZ; that is, again by the Hypothesis of IM to PZ; and the Proportion of the Pyramids CR, FK is triplicate to the Proportion of XQ to ST; that is, again of IM to PZ. Seeing therefore the Proportion of each to each is triplicate to the Proportion of IM to PZ, the Proportion also of all to all (that is, the Proportion of the whole Pyramid AECR to the whole OEFK) will be triplicate to the Proportion of IM to PZ. *Q. E. D.*

PROP. IX. Theorem.

Fig. 18, 19.

EQual Pyramids have their Bases and Altitudes reciprocally proportionable; and those which have them so, are Equal.

Part I. First let the Pyramids be triangular BACO, KHNL: The Parallelograms BE, HR, being perfected, upon these set the parallelopipeds, BF, HP. These will be (as was shew'd in the foregoing) Sixfold of Pyramids which are by the Hypothesis equal, and consequently will be equal betwixt themselves. But now the Altitudes of these Parallelopipeds HK, BA, are the same with those of the Pyramids; and the Bases BE, HR, are double to (*per* 34. *l.* 11.) the pyramidal Bases BCO, HNL, and consequently proportional to them. Seeing therefore by Reason of the Equality of the parallelopipeds, as BE is to HR, so (by the same) is reciprocally HK, to BA; it will also be that as the Base BCO is to the Base HNL, so reciprocally is the Altitude HK to the Altitude BA. *Q. E. D.*

But if the Pyramids have polygonal Bases, let them be reduced into triangular ones, retaining the same Altitudes; and these will be equal to those by the 6th. But the Pyramids thus reduc'd, have, as we have now demonstrated, their Bases and Altitudes reciprocally proportionable. Therefore the given polygonal Pyramids

also have their Bases and Altitudes reciprocally proportional. *Q. E. D.*

Part II. Because it is now suppos'd, that BCO is to HLN, as HK is to BA; BE will also be to HR, as HK is to BA. Therefore the Parallelopipeds BF, HP, are (*per* 34. *l.* 11.) also equal. Therefore their sixth Parts also, to wit, the Pyramids BACO, HKNL are equal. *Q. E. D.*

Corollaries.

WHAT has been demonstrated of Pyramids in *pr.* 6, 8, 9. does also agree to all Prisms whatsoever; seeing these are (*per* 7. *l.* 12.) treble to Pyramids which have the same Bases and Altitudes. Therefore,

1. In Prisms of the same Heighth their Proportion is the same as that of their Bases. For this was shew'd of Pyramids, *pr.* 6.

2. The Proportion of like Prisms is triplicate to the Proportion of their homologous Sides. For this was shew'd concerning Pyramids, *pr.* 8.

3. Equal Prisms have their Bases and Altitudes reciprocally proportionable; and those which have them so are equal. For this is shew'd of Pyramids, *pr.* 9.

It is strange that these Things were pass'd over by *Euclid*, seeing they are the chief Things which can be delivered concerning rectilineal Solids.

Scholium.

FROM what has been hitherto demonstrated is drawn the Method of measuring any Prisms or Pyramids whatsoever.

The Solidity of a Prism is produc'd from the Altitude multiplied into the Base; and that of a Pyramid from the third Part of the Altitude multiplied by the Base.

As if the Altitude of a Prism be of 5 Feet, but the Base contains 25 square Feet; multiply 25 by 5, and there arise 125 cubick Feet for the Solidity of the Prism.

For

For let there be a polygonal Prism as A H. And let the Triangle B A C be understood to be equal to its Base A E, and upon B A C the Prism B E to be set at equal Heighth with A H. The Prisms B E, A H, will be (by *Coroll.* 1. foregoing) equal. But the Prism B E (by *Schol. p.* 40. *l.* 11.) is produc'd from its Altitude drawn into the Base B A C; that is, into A E, by Construction. Therefore the Prism A H also is made of its Base A E multiplied by its Heighth, which is suppos'd to be equal to the Heighth of the Prism B E. *Fig.* 15, 14.

From hence and from 7. the Demonstration of the second Part is also manifest.

A Lemma to Prop. 10.

PYramids and Prisms which are inscrib'd in Cones and Cylinders infinitely, do at last end in the Cones and Cylinders.

This is demonstrated as the *Lemma* of *Prop.* 2. with the help of *Prop.* 6. and of *Coroll.* 2. after *Prop.* 9. if as there Plains inscrib'd in a Circle, so here Prisms and Pyramids which stand upon those Plains as their Bases, be continually taken away from the Cones and Cylinders.

PROP. X. Theorem.

EVery Cone is a third Part of a Cylinder having the same Base and Heighth. *Fig.* 20.

Let a regular Polygon of as many Sides as you please be understood to be inscrib'd in the Base C L, and upon it as the Base, for a Cone let a Pyramid, and for a Cylinder a Prism be inscrib'd. The Pyramid (*per* 7. *l.* 12.) will be a third Part of the Prism. And if again in the Circle a Polygon of twice as many Sides be inscrib'd, and upon it be inscrib'd for a Cone a Pyramid, but for the Cylinder a Prism; the Pyramid will again be a third Part of the Prism. And thus it will always be. Wherefore seeing Pyramids end in a Cone, and Prisms in a Cylinder, the Cone also will be a third Part of the Cylinder. *Q. E. D.*

PROP. XI. Theorem.

Fig. 20, 21. *CONes of equal Height (BAF, QXR) are as their Bases (CL, SE). The same Thing belongs to Cylinders of equal Heighth also.*

Pyramids inscrib'd into Cones of equal Heighth, are as their (*per* 6. *l.* 12.) Bases. But Pyramids do at length end in Cones. Therefore Cones also are as their Bases. And seeing Cylinders are threefold of Cones, which have the same Base and Altitude with them, they also will be as their Bases. *Q. E. D.*

Coroll.

IN the same manner it may be demonstrated that also Prisms and Cylinders of equal Height are betwixt themselves as their Bases; yea, that all cylindrical Bodies of the same Altitude; that is, which are produc'd from whatsoever Plains multiplied by the same Altitude, are betwixt themselves as their Bases. You may reason in the same manner of Pyramids and Cones of equal Altitude, and of all conical Bodies whatsoever.

PROP. XII. Theorem.

Fig. 20, 21. *THE Proportion of like Cones (BAF and QZR) is triplicate to the Proportion of the Diameters (BF and QR) which are in the Bases. The same Thing is to be said of like Cylinders.*

In the Bases of the like Cones let regular Polygons be inscribed, which Polygons consequently will be like. The Pyramids which are inscribed upon these Polygons will also be like; as may be easily shew'd. Therefore their Proportion is triplicate (*per.* 8. *l.* 12.) to the Proportion of the Sides BL, QE; that is, to the Proportion of the Diameters BF, QR. Wherefore seeing

ing the Pyramids end in Cones, the Proportion also of the Cones is triplicate to the Proportion of the Diameters BF, QR. *Q. E. D.*

The Theorem is manifest of Cylinders, seeing they are treble to Cones.

PROP. XIII. Theorem.

IF a Cylinder (BI) be cut with a Plain (RL) parallel to the Bases (BQ, CI); one Part (BL) shall be to the other Part (RI), as one Segment of the Axis (AO) is to the other Segment of the Axis (OF.) Fig. 22.

This Proposition is demonstrated as the first of *l.* 6.

The Theorem is in the same manner true of the Superficies.

PROP. XIV. Theorem.

CYlinders (AR and CI) of equal Bases (MQ, GH) are as their Altitudes (LZ, SF.) The same Thing happens to Cones. Fig. 23, 24.

Cut off from the higher Cylinder AR the Cylinder AO, whose Heighth LE is the same with SF. Therefore (*per* 11. *l.* 12.) the Cylinders AO, CI, are equal. Seeing therefore the Cylinder AO, is to the Cylinder AR, (by the foregoing) as LE is to LZ; CI also shall be to AR as LE is to LZ; that is, (because LE and SF are equal, by Construction) as SF to LZ. *Q. E. D.*

Corollary.

THE Theorem is also true of Prisms, and likewise of Pyramids, and the Demonstration altogether alike. But of Prisms the thing is demonstrated from *Corol.* 1. *p.* 9. *l.* 12, and 25. *l.* 11. and its *Corol.* Of Pyramids from this, and from *p.* 7. *l.* 12.

PROP. XV. Theorem.

Fig. 24, 25. *Equal Cylinders (AR, DF) have their Bases and Altitudes reciprocally proportional; and if they have them so they are equal. The same Thing is true of Cones.*

This is demonstrated as *Prop.* 34. *l.* 11. only for 32. and 25. *l.* 11. there cited, there must be cited here *Prop.* 11, and 13. *l.* 12.

Scholium.

WHereas *Euclid* hath said nothing of compound Proportion in Bodies, we shall briefly demonstrate it in this Place.

1. A Cylinder hath to a Cylinder, and a Prism to a Prism, a Proportion compounded of the Proportions of the Bases and Altitudes.

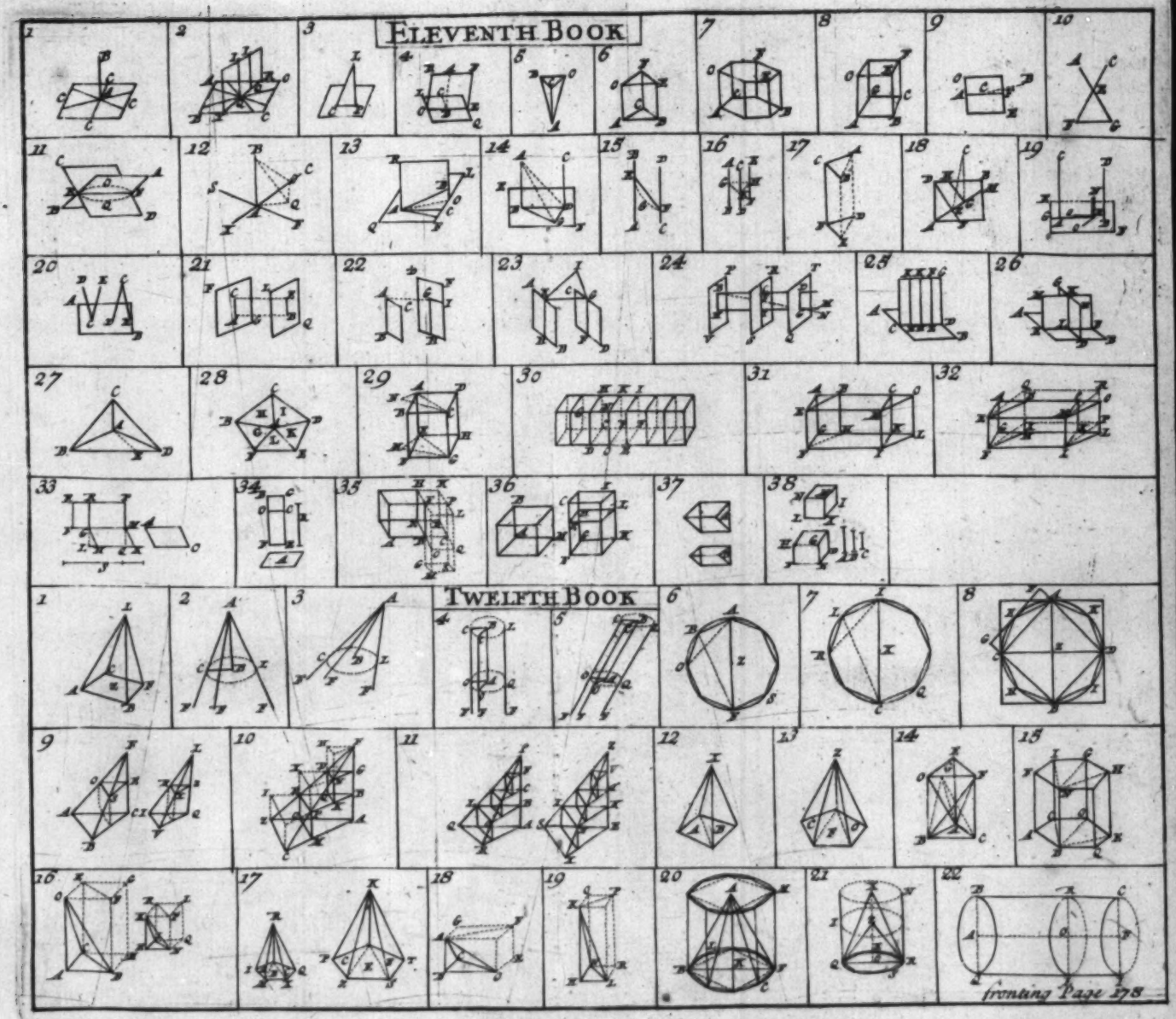
Fig. 25, 24. Let FD and AR be Cylinders of different Altitudes (for in those of equal Altitude the Thing is manifest.) From the higher cut off AO of equal Height with FD. And let the Proportion be thus; as the base UT is to the Base MQ, so FN to X; and as the Altitude ND or BO is to the Altitude BR, so is X to Z. We must therefore shew, that the Cylinder FD is to the Cylinder AR, as FN is to Z. The Cylinder FD is to the Cylinder AO (*per* 11. *l.* 12.) as the Base VT is to the Base MQ; that is, (by Construction) as FN is to X; but the Cylinder AO is to the Cylinder AR (*per* 13. *l.* 12.) as BO to BR; that is, (by Construction) as X to Z. Therefore by Proportion of Equality the Cylinder FD is to the Cylinder AR, as FN to Z.

The Proposition may be demonstrated in the same manner of Prisms, but from *Coroll.* 1. *pr.* 9. and *Coroll. pr.* 14.

2. A Cone also hath to a Cone, and a Pyramid to a Pyramid, a Proportion which is compounded of the Proportions of Base to Base, and Altitude to Altitude.

For (by *Prop.* 10. and 7. *l.* 12.) they are third Parts of Cylinders and Prisms.

PROP.



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PROP. XVI, XVII.

THese Propositions, the most prolix of all other, have no other Use than to serve to the demonstrating, Prop. 18. *which we shall demonstrate in another more easy Way.*

Lemma to Prop. 18.

CYlinders inscrib'd in an Hemisphere end in the Hemisphere. Let PZY be the greatest Semicircle of the Hemisphere; and let the Radius AZ be perpendicular to the Diameter PY. Cut AZ into a certain Number of equal Parts, AM, MN, NZ; and there being drawn through the Points of the Divisions M, N, the perpendicular Lines BO, *&c.* Let there be inscrib'd in the semicircle the Rectangles OBRK, EDHS; which aftewards being continued without the Semicircle, let there be understood to be describ'd about the Semicircle the Rectangles FTYP, LVBO, QXDE. They will all of them be of the same Height, and the Excesses of the circumscribed ones above those which are inscrib'd will be the Plains FK, LS, XE, VH, TR, which taken together make the Rectangle FTYP. For because XE is equal to DS, those LS, VH, XE, together will be equal to the Rectangle LB, that is, OR. Wherefore if you add on both Sides the Plains FK, TR, all those FK, LS, XE, VH, TR, taken together will be equal to the Rectangle FTYP. If now the Semicircle with the Rectangles be understood to be turn'd about the Radius AZ, which is in the mean while unmov'd, the inscribed Rectangles EH, OR, will produce Cylinders inscrib'd in the Hemisphere; and the circumscrib'd Rectangles will produce Cylinders circumscrib'd about the Hemisphere, standing one upon another; and as the Excesses of the circumscribed Rectangles above the inscribed ones was the Rectangle FY; so likewise the Excesses of the circumscribed Cylinders above the inscribed ones, will be the Cylinder which is produced from the Rectangle FY. But now the Altitude of *Fig.* 26.

this Cylinder will be made less than any given Height h; and consequently (as is manifest from 13. *l.* 12.) it self will grow to be less than any given Cylinder, if, the Radius being divided into more equal Parts without end, the Number of Rectangles, and from thence of Cylinders be infinitely increas'd. Therefore the Excess of the circumscrib'd Cylinders, and much more of the Hemisphere it self, which is only a Part of the circumscrib'd Cylinders, above the inscribed ones, will at last become less than any given one. Therefore (by *Defin.* 6. *l.* 12.) Cylinders infinitely inscribed in an Hemisphere, do at length end in the Hemisphere it elf. *Q. E. D.*

Corollary.

IN the same manner it will be demonstrated, that Cylinders inscrib'd in a Cone, Conoid, Spheroid, *&c.* do at last end in the same.

PROP. XVIII. Theorem.

Fig. 27. THE *Proportion of Spheres is triplicate to the Proportion of their Diameters* (*BK, RZ*).

The Radius's AB, YR, being divided into as many equal Parts as you will, but of an equal Number, and there being drawn through the Points of the Divisions perpendicular, *&c.* Let Rectangles of an equal Number be understood to be inscribed in the greatest Semicircles of the Spheres, which Rectangles being turned about the unmov'd Radius's AB, YR will be conceiv'd to inscribe in both the Hemispheres a like Number of Cylinders standing one upon another. Now because KC is (*per Coroll. p.* 13. *l.* 6.) to CF, as CF is to CB; the Proportion of KC to CB (by *Defin.* 10. *l.* 5.) will be duplicate to that of KC to CF, that is, to the Proportion of FC to CB. In like manner the Proportion of ZE to ER will be duplicate to the Proportion of XE to ER. But by the Construction KC is to CB as ZE is to ER. Therefore FC also is to BC as XE to ER. But BC by the Construction is to CO as

RE

RE to ES. Therefore by Equality FC is to CO, as XE is to ES. Therefore (by *Defin.* 4. *l.* 12.) the Cylinders FL, XQ, are like, and consequently their Proportion is triplicate to (*per* 12. *l.* 12.) the Proportion of their Diameters, FI, XV, or of the Semi-diameters FC, XE, which are the Bases. But the Proportion of FC to XE is the same with the Proportion which is betwixt the Diameters of the Spheres BK, RZ; (for as I have already shew'd FC is to XE as CO is to ES; that is, as BK is to RZ which by the Construction are equi-multiples of those CO, ES.) Therefore the Proportion of the Cylinders FL, XQ is triplicate to the Proportion of the Diameters BK, RZ. In the same manner we might demonstrate that each Cylinder inscribed in one Hemisphere, bears to each Cylinder inscribed in the other Hemisphere a Proportion triplicate to the Proportion of the Diameters BK, RZ. Therefore also the Proportion of all together to all together is triplicate to the Proportion of the Diameters BK, RZ. Wherefore seeing the aggregates of the Cylinders do at length end in their Hemispheres, the Proportion of the Hemispheres also, and consequently of the Spheres will be triplicate to the Proportion of their Diameters. *Q. E. D.*

Corollary.

THerefore the Proportion of the Diameters being known, the Proportion of the Spheres becomes known likewise. As if the Diameter of the lesser be one Foot, that of the greater Ten Feet; let the Proportion of one to ten be continued through four Terms, 1, 10, 100, 1000. as 1 the first is to 1000, the 4th Term, so is the lesser Sphere to the greater.

The Dimension of Cones, Cylinders, and of the Sphere, will be exhibited in the following Book out of *Archemides*.

Scholium.

AS like plain Figures are increas'd or diminished in any given Proportion by one mean Proportional, so like

like Bodies are increas'd or diminish'd by two mean Proportionals.

Let a Sphere or Cube, or any other Body whatsoever be given, whose Radius or Side is A. Likewise let any Proportion whatsoever of A to B be given, as the double or 2 to 1. A Body is to be discover'd both double to the given one, and like to it.

Betwixt the Terms of the given Proportion A and B, let there be found two mean Proportionals X, Z, according to what was taught in the *Scholium* of *Prop.* 13. *l.* 6. A Sphere whose Radius is X, or other Body like to the given one which is made upon the Side X, will be double to the given one.

For like Bodies whose Radius's or Sides are A and X, have betwixt themselves a Proportion which is triplicate to the Proportion of A to X, (by *Coroll. prop.* 9, and by *Prop.* 12. and 18. *l.* 12.) that is, the same (*per Defin.* 10. *l.* 5.) which A hath to B.

And this is that most celebrated Problem which from *Apollo* of *Delos* is called the *Deliacal* Problem; because at the time of almost grievous Pestilence, which wasted *Athens*, being consulted, he gave answer, that the Pestilence would cease, if his Altar, which was of a cubical Form, were doubled. Thus *Valerius Maximus* l. 8.

THE-

THEOREMS
Selected out of
ARCHIMEDES:
By ANDREW TACQUET,
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To which are added,
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LONDON:
Printed in the Year MDCCXIV.

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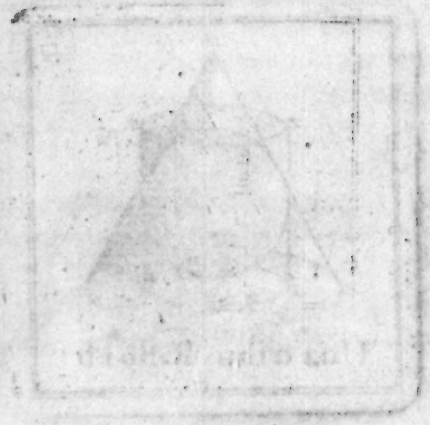

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To the READER.

ALBEIT *there have appear'd very many most excellent and admirable Men in the Mathematical Sciences; yet the chief Glory of all hath always by a certain common Consent, been given to* Archimedes *of* Syracuse. *Tho' indeed more there are who commend than who read him; more who admire than understand him. The Causes of which Neglect seem to be these, the Bulk and Scarceness of Copies, some Obscurity of the Translation, which is directly made out of the* Greek *Language, together with the Prolixity and Difficulty of his Demonstrations. I judged therefore that it would be for the Profit of studious Learners, if after my Illustration of the Elements, I should subjoin these* Theorems *which have been selected by me out of* Archimedes, *and demonstrated in a much easier and briefer Way. Furthermore, I have selected those, which bring along with them both more of Admiration and of Benefit; and have in my Demonstration took such a Method that, I hope, he who understands the Elements will, without any great Labour comprehend these most excellent Inventions of the Prince of Geometricians. I have also added at the End thirteen Propositions, and thereby enlarged the Doctrine of* Archimedes *concerning the Sphere and Cylinder: where amongst other Things, I demonstrate, that the sesquialteral Proportion is continued in the Three Bodies, a Sphere, Cylinder, and equilateral Cone; both the latter be-*

ing circumscrib'd about the Sphere. Moreover I have added divers Things here and there, amongst which the 12*th* Proposition, *and the* Corollaries *of* Prop. 14. *are the chief; and several* Scholiums. *Make use of these Discoveries whosoever thou be'st, that art a Candidate of* Geometry; *and how much thou hast improv'd in* Euclid, *make Proof of in* Archimedes. *And when thou perceivest thy self to be fix'd and rais'd upwards in the Contemplation of the most noble Truths, raise up thy Mind, while it is thus already lifted up from these lower Things yet higher, and direct it to that Truth which is Original, Eternal, Immense, and is no other than God; by the ineffable Vision of whom, I trust we shall hereafter be made eternally Happy. Farewel.*

THEO-

THEOREMS
Selected out of *ARCHIMEDES.*

DEFINITIONS;
Or an Explication of certain Terms.

LET there be a Circle BECG, whose Center is A, its Diameter BC, which let the right Line EG cut at right Angles, but not through the Center, in D. Let there be drawn from the Center the Radius's AE, AG. This being suppos'd, *Fig.* 23. Of the Table out of *Archimedes.*

NOTE, (1.) That a Sector of a Sphere is that which is produced from the Sector of the Circle AECG, or AEBG, turn'd round about the Diameter BC.

2. That a Segment or Portion of a Sphere is that Part of it which is produc'd from the Segment of the Circle ECG or EBG turn'd round about the same Diameter BC.

3. The Vertex or Top of the Spherical Portion EBG is the Extremity B of the unmov'd Diameter; the Basis, the Circle describ'd by EG; the Axis, that Part of the Diameter BD, which is intercepted betwixt the Top B, and D the Center of the Base.

4. When I name the Superficies of a Spherical Portion, or of a Body inscribed in it, or of a Cone, I always understand it without the Base; and when I say the Superficies of a Cylinder, I mean likewise without the Bases; unless the Word [whole] be adjoin'd to [Superficies]; for then the Bases also are to be taken in.

Again,

Again, when I treat of Cylinders or Cones, I speak of no other than right ones.

Axioms.

Fig. 1, 16. 1. THE Circuit of a Polygon inſcrib'd in a Circle is leſs than the Circumference of the Circle.

Fig. 1. 2. The Circuit of a Polygon deſcrib'd about a Circle is greater than the Circumference of the Circle.

Fig. 16. 3. And if a Polygon inſcrib'd in a Circle, be turn'd about the Diameter (A E) together with the Circle, the Superficies of the Body produc'd by the Polygon, will be leſs than the Superficies of the Sphere. And if a Polygon circumſcrib'd about the Circle, be turn'd about the Diameter, together with the Circle, the Superficies of the Body produc'd by the Polygon will be greater than the Superficies of the Sphere.

Fig. 17. 4. In like manner the Circuit of a Polygon inſcrib'd in a Segment of a Circle (D A F) is leſs than the Circumference of the Segment. And if a Polygon inſcrib'd in the Segment, be together with the Segment (A O) turned round; the Superficies of the Body produc'd by the Polygon will be leſs than the Superficies of the Spherical Portion (D A F.)

Fig. 3, 6. 5. The Superficies of a Priſm inſcrib'd in a Cylinder is leſs than the Superficies of the Cylinder; but the Superficies of the Priſm which is circumſcrib'd is greater.

Fig. 4, 8. 6. And the Superficies of a Pyramid inſcrib'd in a Cone, is leſs than the Superficies of the Cone; but the Superficies of a circumſcrib'd Pyramid, is greater.

PROPOSITIONS I, II.

ARE *not neceſſary.*

PROP. III. Theorem.

THE *Circuits of Polygons circumſcrib'd about and inſcrib'd in a Circle do at laſt end in the Circumference of the Circle. In lik manner the*

the Polygons themselves do at last end in the Circle.

If, to wit, the Arches being bisected without end; more and more Sides be circumscrib'd about and inscrib'd in the Circle. *Fig. 1. Tabl. Archimed.*

Part I. Let there be understood to be inscrib'd in and describ'd about a Circle regular Polygons; whether it be done so as is set down, *pr.* 12. *l.* 4. or as in the present Figure, the Thing will be the same. It is manifest (*per Coroll.* 1. *p.* 4. *l.* 6.) that F I is to C E (that is, the whole Circuit circumscrib'd, unto the whole Circuit inscrib'd) as I A is to C A. But I C the Excess of the right Line I A above C A, becomes at length less than any given Line, if more and more Sides be understood to be infinitely circumscrib'd and inscrib'd; therefore also the Excess of the Circuit circumscrib'd above that which is inscrib'd, will at length become less than any given Line. Therefore much more the Excess of the Circuit circumscrib'd above the Circumference of the Circle will be less than any given one. In like manner, because I have already shew'd the Defect of the Circuit inscrib'd, whereby it falls short of that which is circumscrib'd, to be less than any given Line: Therefore much more will the Defect of the Circuit inscribed, whereby it falls short of the Circumference of that Circle, become less than any given Line. The Circuits therefore, as well as that which is inscrib'd, as that which is circumscrib'd, do at length (*Defin.* 6. *l.* 12.) end in the Circumference. Which was the first Part. To demonstrate these Things further is not worth the while, seeing they are manifest enough.

Part II. Because it hath already been shew'd that the Excess F I above the Side E C becomes at length less than any given (for F I is to E C, as I A to C A); therefore also the Excess of the Square of F I above the Square of E C will become at length less than any given. But as the Square of F I is to the Square of E C so (*per* 20. *l.* 6.) is the Polygon circumscrib'd, to that which is inscrib'd. Therefore the Excess of the Polygon circumscrib'd above that which is inscrib'd, will also become at length less than any given one. Therefore much more will the Excess of the Polygon circumscrib'd above the Circle, become at last less less than any given one;

and consequently, the Defect also of the Polygon inscrib'd, whereby it falls short of the Circle, will at length become less than any given Defect. Therefore Polygons as well inscrib'd as circumscrib'd do at last (*Defin. 6. l. 12.*) end in the Circle. Which was the second Part.

PROP. IV. Theorem.

(c) *Per def. 3. l. 4.* *Fig. 1.*

A Regular (c) *Polygon (FINTR) circumscrib'd about a Circle, is equal to a Triangle whose Base is the Circuit of the Polygon, and it's Heighth the Radius of the Circle.*

And a regular Polygon inscrib'd in a Circle is equal to a Triangle, which hath for its Base the Circuit of the Polygon, and for its Heighth the Perpendicular (AO) let down into one Side from the Center.

Part I. The Radius AB drawn to the Point of Contact is (*per* 18 *l.* 3.) perpendicular to the Tangent IF. Wherefore if the right Lines AF, AI, AN, *&c.* being drawn, the Polygon be resolv'd into Triangles; the Radius AB will be the common Altitude of all; and consequently it is manifest that the Triangles are equal. Therefore a Triangle which hath its Base equal to the Sum of the Sides FI, IN, NT, *&c.* and AB for its Altitude, will (as is manifest from 1. *l.* 6.) be equal to them all, that is, to the whole Polygon circumscrib'd.

Part II. This may be concluded by the same reasoning as the other.

PROP. V. Theorem.

Fig. 2.

A Circle is equal to a Triangle, which hath for its Base the Circumference, and for its Heighth the Semidiameter of the Circle.

Regular Polygons circumscrib'd about a Circle, and Triangles which have for their Bases the Circuit of the Po-

Polygon, and for their Altitude the Radius of the Circle, are always (by the foregoing *Prop.*) equal. But Polygons circumscrib'd infinitely about the Circle end in the Circle, (by the 3d of this Book); and in like manner Triangles (as I will shew by and by) which have for their Base the Circuit of the circumscribed Polygon, and for their Altitude the Radius A B, at last end in a Triangle which hath the Circumfrence for its Base, and for its Altitude the Radius A B. Therefore (by the first) a Circle and a Triangle which hath the Circumference for its Base, and the Radius for its Altitude are equal.

But that Triangles contain'd under the Circuit of the Polygon, and the Radius of the Circle, end at last in a Triangle, which is contain'd under the Circumference and the Radius, I thus shew. Triangles under the Circuit of the circumscribed Polygon and the Radius A B, are to the Triangle which is under the Circumference and the Radius A B (by 1. *l.* 6.) as Base to Base, that is, as the Circuit of the Polygon to the Circumference; since this Triangle and the other have a common Altitude. But the Circuit of the Polygon (by the 3d) ends in the Circumference. Therefore the other Triangles end in this.

Corollaries.

1. FRom this and 41. *l.* 1. it is manifest that a Rectangle under the Radius and half the Circumference is equal to the Circle; that one under the Radius and the whole Circumference is double; that one under the whole Circumference and whole Diameter is quadruple thereto.

2. A Circle is to an inscribed Square, as half the Circumference (C D E) is to the Diameter; but to a Square circumscribed, as the fourth Part of the Circumference is to the Diameter. *Fig.* 5. *l.* 4.

For the Rectangle under C D E, and the Radius C A or C F that is, (by the foregoing *Corollary*) the whole Circle, is to the Rectangle G F C E, to wit, the Rectangle under F G and C F (that is, to the inscribed Square C D E) as (*per* 1. *l.* 6.) C D E, half the Circumference, is to F G or C E the Diamether; which was the first Thing. And consequently the Circle is to the dou-

ble the Rectangle GFCE, (that is, to FH the circumscribed Square) as CDE is to the double of the Diameter CE, or as the quadrant CD is to the Diameter CE.

Fig. 30. [3. *Of Figures which are of equal Circumferences the Circle is the most capacious. Let the Circumference of any Polygon whatsoever (as for Instance of a Square) EGHI be equal to the Circumference of the Circle. I say, that the Area of the Circle is greater than that of the Polygon. For the Area of the Circle is equal to a Triangle, whose Base is the Circumference, and its Altitude the Semidiameter FA: And the Area of the Polygon is equal to a Triangle whose Base is the Compass of the Polygon, which by the Hypothesis is equal to the Circumference of the Circle, and which hath for its Altitude the Perpendicular FO, let down from the Center of the Circle unto the Side of the Polygon; which since it is always less than the Radius of the Circle, it is manifest that the Area of the Polygon is less than the Area of the Circle.* Q. E. D. *And in like manner, amongst all solid Figures contain'd under equal Surfaces, the Sphere may be demonstrated to be the most capacious.*]

PROP. VI. Theorem.

THE *Circumference of a Circle contains the Diameter less than thrice and one seventh* (*or* $\frac{10}{70}$); *and more than thrice and* $\frac{10}{71}$.

For the Demonstration of this Theorem, *Archimedes* assumes regular Polygons, one circumscribed about a Circle, the other inscrib'd, and both of them of 96 Sides. And then he shews that the 96 inscribed about a Circle, do contain the Diameter less than thrice and one seventh, and consequently that the Circumference which is less than them, doth also contain the Diameter less than thrice and one seventh. But the 96 Sides inscribed in the Circumference, (and consequently the Circumference also which is greater than them) doth contain the Diameter more than three times $\frac{1}{7}$. But this Demonstration is too long to be brought in in this Place. Nay, if we be minded to extend our Geometrical Reasoning to Polygons of more Sides still, we may contract the Limits even now set more and more without Limit,

and

and so come nearer and nearer for ever to the true Proportion. This hath been perform'd by *Ludolph Ceulen*, *Grimberger*, *Metius*, *Snellius*, and others. The chief Proportions hitherto found I shall here subjoin.

[*However, since a Tangent of* 30 *Degrees multiplied by* 12, *gives the Circuit of a circumscribed Hexagon*; *and a Sine of* 30 *Degrees multiplied by* 12 *gives the Circuit of an Hexagon, which is inscribed: Forasmuch also as in like mannen the Tangent of half a Degree multiply'd by* 720, *yields the Circuit of a circumscribed Polygon of* 360 *Sides*; *and the Sine of half a Degree, the Circuit of an inscribed Polygon of* 360 *Sides*; *and so on for ever: It will not be difficult to understand, by what Means many such Numbers may be found, out of the now given Tables of Sines and Tangents.*]

The first Proportion, which is that of *Archimedes*, is thus:

The Diameter 7
The Circumf. is 22; which is greater than the true.
The Diameter 71
The Circumf. is 223; less than the true one.

The Proportions of 22 to 7, and 223 to 71, if they be reduced to a common Consequent, (which is done after the same manner, in which Fractions are reduced to the same Denomination) will be thus, 1562 to 497, and 1561 to 497.

Therefore the Diameter being suppos'd 497 Parts, the Circumference greater than the true one will be 1562; and the Circumference less than the true 1561.

Both of them therefore differ from the true, by a Quantity less than $\frac{1}{497}$ Part of the Diameter. But if the Proportions of 7 to 22, and 71 to 223 be reduced to a common Consequent, there will arise the Proportions of 1561 to 4906, and of 1562 to 4906.

Therefore the Circumference being suppos'd to be 4906 Parts, the Diameter less than the true will be 1561, the Diameter greater than the true 1562.

Both therefore differ from the true Diameter by Quantity less than $\frac{1}{4906}$ Part of the Circumference.

The Proportion delived by *Metius* is much more accurate than this of *Archimedes*. According to this,

The Diameter is 113.
The Circumference 355.

Amongst all Proportions consisting of small Numbers none comes nearer to the true one; for from this, the Diameter being suppos'd of 10,000,000 Parts, the Circumference comes to be of 31,415,929, which differs from the true one only in the first Figure 9, and this by an excess but a little greater than two ten-millioneth Parts of the Diameter.

But more exact than both is that double Proportion of *Ludolphus a Ceulen*; the former of which consists of 21 Figures, and the latter of 36.

The Diameter
100,000,000,000,000,000,000.
The Circumf. greater than the true
314,159265,358979,323847.
The Circumf. less than the true
314,159265,358979,323846.

The Difference of both the Circumferences is one Particle of the Diameter denominated from a Number which consists of a Unity and 20 Ciphers; and consequently as well this as that differs from the true Circumference by a Quantity less than is the said small Part of the Diameter; to wit, one hundredth of a millioneth of a millioneth of a millioneth Part.

The Diameter
100000,000000,000000,000000,000000,000000.
The Circumf. greater than the true
314159,265358,979323,846264,338327,950289.
The Circumf. less than the true
314159,265358,979323,846264,338327,950288.

The Difference of both the Circumferences betwixt which is the true one, is that small Part of the Diameter, denominated from a Number which consists of Unity and 35 Cyphers; which small Part bears a less Proportion to the whole Diameter, than one little Grain of Sand doth to thewhole Globe of the Earth. For the whole Globe of the Earth doth not consist of so many little Grains of Sand, as are the little Parts of the said Sort which are contain'd in the Diameter.

It is needless to go any further. Nevertheless you may proceed infinitely, if you be minded to continue Geometrical Reasoning, an expedite Method of which is delivered by *Snellius*. [*The*

[*The Circumference being suppos'd of*
1,000000,000000,000000,000000,000000,000000,000000 *Parts.*
The Diameter will be as near as may be, of
0,318309,886183,790671,537767,926745,028724 *Parts.*]

Scholium.

THE most excellent Advantages of the Proportion now delivered are these which follow.

The Invention of the Diameter from the Circumference.

SET the greater Term of one of the Proportions which have been now delivered in the first Place, the lesser in the Second, the Circumference in the Third; by these three Numbers let there be sought by the golden Rule a Fourth Proportional. That is the Diameter sought.

As if the Circumference of the greatest Circle of the Earth be suppos'd to contain 25000 *English* Miles of 5280 Feet each, and the Diameter be sought; the Terms will stand thus,

355——113——25000——7854

Multiply now the second by the third, and divide the Product by the first; and there will arise 7854 Miles for the Diameter of the Globe of the Earth.

The finding out of the Circumference from the Diameter.

LET the lesser Term of one of the Proportions above delivered be set in the first Place; the greater in the second; the known Diameter in the third: and by these three Numbers let there be sought a fourth Proportional. That will give the sought Circumference.

As if the Diameter of the Globe of the Earth be suppos'd to contain 7854 *English* Miles; and the Circuit is sought; the Terms will stand thus.

113——355——7854——25000.

Then multiply the second by the third, and divide the Product by the first; there will arise 25000 Miles for the Circumference of the Globe of the Earth.

How little this Circumference exceeds the true one was said above; to wit, by an excess but a little greater than are two ten-millioneth Particles of the Earth's Diameter; that is, by 9 or 10 Feet. But if we use the *Ludolphin* Proportion, even the former, the Terms whereof consist of 21 Figures; there will be found a Circumference insensibly differing from the true, not only when the given Diameter is of 7854 Miles, such as is the Diameter of the Earth; but also although the Diameter be suppos'd of an 100 Millions of those Miles. For this being suppos'd, there will arise a Circumference differing from the true one by a Quantity about one hundred-millioneth Part of a Foot. But if to find out the Circumference of the Globe of the Earth we make use of the Proportion of *Archimedes*, the Interval of the two Circumferences the one greater, the other less than the true one, will exceed 20 Miles. *Archimedes* his Proportion therefore is not to be used but in small Measures; nay, it will always be expedient to use that of *Metius*, which both consists of small Terms, and is above a 1000 times more exact.

The measuring of a Circle.

THE Semidiameter multiplied by half the Circumference produceth the Area of the Circle; as is manifest from *Corol.* 1. *Prop.* 5. of this Book.

As if the Semidiameter of theEarth, which contains 3926 Miles, be multiplied by half its Circumference, to wit, by 12500, there will arise 49,075000 Miles square for the Area of the greatest Circle of the Earth. The Difference of the circular Area thus found from the true is had, if the Difference of half this found Circumference from the true half-Circumference be multipled by the given Semidiameter; or the difference of this Semidiameter from the true, be multiplied by the given Semicircumference.

The Mensuration of Cylinders and Cones.

I Put this here, because it depends upon the Mensuration of a Circle. A Cylinder therefore, and any Prism what-

whatſoever is produced from the Altitude multiplied by the Baſe: A Cone and Pyramid from the third Part of the Altitude multiplied by the Baſe; for they are third Parts of Cylinders and Priſms, having the ſame Baſe and Altitude with them, by 10, and 7, *l.* 12.

Let the Baſe of a Cylinder or Cone be of 50 ſquare Feet, and the Height of 100 Feet. Multiply 100 by 50, and there ariſe 5000 cubick Feet for the Solidity of the Cylinder. Multiply the third Part of the Altitude 100, which is $33\frac{1}{3}$ by 50, there ariſe $1666\frac{2}{3}$ cubick Feet for the Solidity of the Cone.

PROP. VII. Theorem.

THE Circumferences of Circles have the ſame Proportion betwixt themſelves, which their Diameters have. Fig. 6. l. 12.

For the Circuits of like Polygons, which may be inſcribed in a Circle without end, are always betwixt themſelves as the Diameters A F and I C (by *Coroll. pr.* 1. *l.* 12.) But theſe Circuits (by the third *pr.* of this Book) end at length in the Circumference. Therefore their Circumferences alſo are betwixt themſelves as their Diameters. *Q. E. D.*

PROP. VIII. Theorem.

THE Superficies of a Priſm, as well that which is circumſcrib'd about, as that which is inſcrib'd in a Cylinder, is equal to a Rectangle whoſe Heighth is the Side of the Cylinder, but its Baſe equal to the Circuit of the Baſe of the Priſm.

Part I. The Superficies of the circumſcribed Priſm touches the Cylinder according to the Lines E A, N F, *&c.* which are the Sides of the Cylinder; but theſe (becauſe by the Hypotheſis the Cylinder is a right one) are right to the Plain of the Baſe, and conſequently right alſo (by *Defin.* 3. *l.* 11.) to the Lines C G, G M, *&c.* But they are alſo equal betwixt themſelves. Therefore Fig. 3.

one

one Side of the Cylinder is the common Heighth of all the Rectangles CO, OM, MH, *&c.* Therefore the Superficies of the circumscribed Prism is equal (as is manifest from 1. *l.* 6.) to a Rectangle contain'd under the Circuit of the Base of the Prism, and the Side of the Prism or Cylinder.

Part II. The Reason of this is the same. For the Side of the Cylinder is again the common Altitude of the Rectangles BDIK, KIQP, *&c.* which constitute the Superficies of the inscribed Prism.

PROP. IX. Theorem.

Pr. 4. *THE Superficies of a regular Pyramid circumscrib'd about a right Cone, is equal to a Triangle, which hath for its Base the Circumference (FHLD) of the pyramidal Base, but its Height the Side of the Cone (BG.)*

And the Superficies of a regular Pyramid inscribed in a right Cone, is equal to a Triangle, which hath for its Base the Circumference of the pyramidal Base, but for its Height the perpendicular (BO) let down from the Top unto a Side of the Base.

Part I. Let there be drawn unto the Contacts, G, K, M, the right Lines BG, BK, BM. These will all be Sides of a right Cone, and consequently equal. And, because (by the Hypothesis) the Axis BA is perpendicular to the Plain of the Base FKD, the Plain also GBA (*per* 18. *l.* 11.) will be perpendicular to the Plain FKD. But HG (*per* 18. *l.* 3.) is perpendicular to AG the common Section of the Plains FKD and GBA. Therefore HG (as is gathered from *Defin.* 4. *l.* 11.) is also perpendicular to the Plain GBA. And consequently is also perpendicular to BG. Therefore the Side GB of the Cone, is the Heighth of the Triangle FBH. In the same manner the Side of the Cone will be the Heighth of the rest HBL, LBD, *&c.* Therefore the Triangle comprehended under the Circumference FH

LD

LD and the Side of the Cone is equal to the Superficies of a Pyramid circumscribed, without the Base. Which was the first Part.

II. The Demonstration of this Part is almost the same with that of the former.

PROP. X. Theorem.

THE Superficies of a regular Prism circumscrib'd about a right Cylinder, ends (Defin. 6. l. 12.) *in the Superficies of the Cylinder; and the Superficies of a Pyramid circumscrib'd about a right Cone, ends in the Superficies of the Cone.*

Part I. The Superficies of regular Prisms described about and inscribed in a Cylinder without end, will have at last a difference betwixt themselves less than any which can be given. Much more therefore will the Superficies of a circumscribed Prism differ from the Superficies of the Cylinder, which is in the middle between the inscribed and circumscribed Superficies, by a Difference less than any given one whatsoever; that is, (*Def.* 6. *l.* 12.) will end in the cylindrical Superficies, whilst it continually exceeds it less and less. *Fig.* 3.

Part II. This may be shewed in the same manner from the 9 and 3 of this. *Fig.* 4.

In the Figures there are only exhibited the halves of the Cylinder and Cone, lest a Multitude of Lines should breed Confusion. But the Cylinder and Cone are to be conceiv'd in the Mind entire, and as having thus circumscribed Prisms and Pyramids encompassing them. For thus it more clearly appears that plain Surfaces circumscribed are greater, according to the 3d *Axiom.*

A Lemma *to the following* Proposition.

LET AB, CD, EF, be proportional, and let KB be half AB, and EG, double EF; KB, CD, EG, will also be proportional. *Fig.* 7.

The right Line KB is to AB as EF is to EG. Therefore the Rectangle KB, EG (*per* 16. *l.* 6.) is equal

qual to the Rectangle AB, EF. But this (by 17. *l.* 6.) is equal to the Square of CD. Therefore also the Rectangle KB, EG, is equal to the Square of CD. Therefore (by 17. *l.* 6.) KB, CD, EG are proportional.

PROP. XI. Theorem.

Fig. 5, 6. *A Circle, whose Radius (GH) is a mean proportional betwixt the Side of a right Cylinder (BC) and the Diameter of the Base (BD) is equal to the cylindrical Superficies.*

Let the regular and consequently like Polygons, NM, RS, be understood to be circumscribed about the Circles ABN, GPH; and upon the Polygon NM let a Prism be conceived to be erected, with which the Cylinder is circumscribed. Because BD, GH, BC are by the Hypothesis proportional, AD also (or AN) GH, and the double of BC will by the *Lemma* be proportional. Now the Triangle contain'd under AN and the Circuit of the Polygon MN is equal to the Polygon circumscribed NM (by the fourth of this Book): And the Rectangle under BC, or EF, and the same Circuit NM (that is, as is manifest from 41. *l.* 1. the Triangle under the Circuit NM, and the double of BC) is equal (by the 8th of this Book) to the Superficies of a Prism circumscrib'd about the Cylinder. But a Triangle under the Circuit NM and AN is to the Triangle under the Circuit NM, and the double of BC (by 1. *l.* 6.) as AN is to the double of BC. Therefore the Polygon NM also is to the Superficies of a Prism circumscribed about a Cylinder, as AN is to the double of BC. But because I have already shew'd AN, GH, and the double of BC to be proportional, the Proportion of AN to the double of BC is (by *Defin.* 10. *l.* 5.) duplicate to the Proportion of AN to GH. Therefore the Polygon NM hath to the Superficies of the Prism a Proportion duplicate to the Proportion of AN to GH. But the Polygon NM hath also to the Polygon like to it GRQS a Proportion duplicate to the Proportion of AN to GH, as is easily gathered out of 1. *l.* 12. Therefore the Polygon NM hath the same Proportion to the Superficies of the

the Prism, which it hath to the Polygon GRQS; which consequently is equal to the Superficies of the Prism. In the same manner, I might shew that the prismatic Superficies, which are circumscriptible infinitely about the Cylinder, are always equal to the Polygons which may be circumscribed infinitely about the Circle GPH. Wherefore seeing both the prismatic Superficies (by the 10th of this) end in the Surface of the Cylinder, and the Polygons in the Circle GPA (by the 3d of this) the Superficies of the Cylinder also will be equal to the Circle GPH. *Q. E. D.*

From this admirable Theorem, a Circle is presented which is equal to a cylindrical Superficies.

Corollaries.

THE Superficies of a right Cylinder is equal to a Rectangle contain'd under the Side (BC) and the Circumference of the Base. Fig. 5, 6.

The double of BC (as hath been shew'd above) is to GH, as GH is to BA, or AN; that is, (by the 7th of this) as the Circumference P is to the Circumference BN. Therefore the Triangle under the first, to wit, the double of BC; and the fourth, to wit, the Circumference BN, is equal to a Triangle under the second GH, and the third, to wit, the Circumference P, (as appears from 16. *l.* 6.) But the Triangle under GH and the Circumference P, is (by the 5th of this) equal to the Circle GPH, that is, (by the 11th of this) to the cylindrical Superficies. Therefore also the Triangle under the double of BC and the Circumference BN, (that is, as appears from 41. *l.* 1. the Rectangle which is under BC and the Circumference BN) will be equal to the cylindrical Superficies. *Q. E. D.*

From this *Corollary* it is manifest, that the Properties of Rectangles are common to them with right cylindrical Superficies. Therefore let this be *Corollary* 2.

2. The cylindrical Superficies (BM, QN) which are of the same Height, are betwixt themselves as the Diameters of their Bases) BF, QR). Fig. 20, 21. *l.* 12.

For the Rectangles under the Circumferences (CL, SE) and the equal right Lines FM, RN, to which (by

(by *Corol.* 1.) the cylindrical Superficies are equal, are betwixt themselves (by 1. *l.* 6.) as the Bases, to wit, the Circumferences CL, SE; that is, as the Diameters BF, QR (by the 7th of this.)

3. The cylindrical Superficies (CI, AR) which have equal Bases are betwixt themselves, as their Altitudes (TI, BR.)

Fig. 23, 24. *l.* 12. For the Rectangles contain'd under the equal Circumferences GH, MQ, and the Sides TI, BR, to which (by *Corol.* 1.) the cylindrical Surfaces are equal, are betwixt themselves (by 1. *l.* 6.) as TI, BR.

Fig. 20, 21. *l.* 12. 4. Like cylindrical Surfaces (BM, RI,) have betwixt themselves a Proportion duplicate to that, which (BF, QR,) the Diameters of the Bases have.

Seeing the Cylinders are suppos'd to be like, MF will be to IQ (by *Defin.* 4. *l.* 12.) as BF is to QR; that is, (by the 7th of this) as the Circumference CL to the Circumference SE. Wherefore the Rectangles also which are contain'd under the Circumferences CL, SE, and the Sides MF, IQ, will be like; and consequently they will have betwixt themselves (by 20. *l.* 6.) a Proportion duplicate to that which MF hath to IQ; that is, BF to QR. Therefore the cylindrical Surfaces also have the same.

The same Figure. 5. Cylindrical Surfaces (BM, RI,) have betwixt themselves a Proportion compounded of the Proportions of the Sides (FM, IQ,) and the Diameters of the Bases (BF, QR,) as is manifest from 23. *l.* 6. and the 7th of this.

Fig. 24, 25. *l.* 12. 6. If cylindrical Surfaces (AR, FD) be equal; as the Diameter (AB) is to the Diameter FN,) so reciprocally (by 14. *l.* 6.) the Altitude (FH) will be to the Altitude (KB;) and conversly.

7. Lastly, from the same 1st *Corol.* is had the Measure of a cylindrical Snperficies; to wit, if the Circumference of the Base be multiplied by the Altitude. As if the Altitude be of 20 Feet, the Circumference of the Base of 6; multiply 20 by 6, there arise 120 square Feet for the cylindrical Superficies.

PROP.

PROP. XII. Theorem.

THE Superficies of a right Cylinder is to the Base (ABN) as the Side of the Cylinder (CB) is to (BO) the fourth Part of the Diameter of the Base. Fig. 6, 5. of this.

Let GH be a mean Proportional betwixt BC the Heighth, and BD the Diameter of the Base, and consequently (by *Lemma* before *Prop.* 11. of this) a mean Proportional betwixt AN and the double of BC. The Circle GPH of the Radius GH is (by the 11th of this) equal to the curve cylindrical Superficies CD. But the Circle GPH hath to the Base of the Cylinder ABN a Proportion duplicate (by 2. *l.* 12.) to the Proportion of GH to AN; that is, the same which the double of BC hath to the Radius BA (by the Hypothesis, and *Def.* 10. *l.* 5.) that is, the same which BC hath to BO, the fourth Part of the Diameter. Therefore the cylindrical Superficies also is to the Base ABN as BC is to BO, the fourth Part of the Diameter. *Q. E. D.*

Corollary.

THE Superficies of a Cylinder which hath its Side equal to the Diameter of its Base, is fourfold of the Base. But if the Side be a fourth Part of the Diameter of the Base, the Superficies of the Cylinder will be equal to the Base. Both these are manifest from the Proposition.

PROP. XIII. Theorem.

A Circle whose Radius (OL) is a mean Proportional betwixt the Side (BC) of a right Cone, and the Radius of the Base (AC) is equal to the conical Superficies. Fig. 9, 8.

Let regular Polygons EF, IN, be understood to be circumscrib'd about the Circles ACG, OPL, and a Pyramid circumscrib'd about the Cone to be erected upon the Polygon EF.

Because, by the Hypothesis, AC, or AG, is to OL, as OL is to BC, the Proportion of AG to BC will (*Defin.* 10. *l.* 5.) be duplicate to the Proportion of AG to OL. But as AG is to BC, so is the Triangle under AG and the Circuit EF to the Triangle under BC and the same Circuit EF. Therefore the Proportion of the Triangle under AG and the Circuit EF, to the Triangle under BC, and the same Circuit, is also duplicate to the Proportion of AG to OL. But the Triangle under AG and the Circuit EF is equal to the Polygon EF (by the 4th of this): And the Triangle under BC and the same Circuit EF (by the 9th of this) is equal to the Superficies of the circumscribed Pyramid. Therefore the Proportion of the Polygon EF to the Superficies of the Pyramid is also duplicate to the Proportion of AG to OL. But the Proportion of the Polygon EF to the Polygon IN, which is by the Construction like to it, is (*per* 1. *l.* 12.) also duplicate to the Proportion of AG to OL. Therefore the Polygon EF hath the same Proportion to the Superficies of the Pyramid, and to the Polygon IN, which consequently are equal. In the same manner I might shew that the Superficies of Pyramids, which may be circumscribed about a Cone infinitely more and more, are always equal to Polygons which may be circumscribed infinitely about the Circle OPL. Wherefore seeing both the Surfaces of Pyramids (by the 10th of this) do at last end in the Surface of the Cone, and Polygons (by the 3d of this) in the Circle OPL, the Superficies of the Cone and the Circle OPL shall likewise be equal. *Q. E. D.*

From this excellent Theorem a Circle is found which is equal to a conical Surface.

Corollaries.

Fig. 9, 8. 1. THE Superficies of a right Cone is equal to a Triangle comprehended under the Side of the Cone (BC) and the Circumference of the Base (CG).

Let OL the Radius be a mean proportional betwixt AC and BC. Then becauſe (by the 7th of this) the Circumference CG is to the Circumference P as the Radius AG is to the Radius OL; that is, by the Hypotheſis, as OL is to BC; the Triangle under the firſt, to wit, the Circumference CG and under the fourth BC (as appears from 16. *l.* 6.) will be equal to the Triangle under the ſecond; to wit, the Circumference P, and the third OL; that is, (by the 5th of this) to the Circle OPL; that is, to the Conical Superficies (by the 13th of this) BCD. *Q. E. D.*

From this *Corollary* it appears that conical Surfaces have the ſame Properties with Triangles. And ſo it follows,

2. That the conical Superficies (BAF, QXR) having their Sides (BA, QX) equal, are betwixt themſelves as the Baſes of their Diameters (BF, QR.) And, *Fig.* 20, 21. *l.* 12.

3. Thoſe which have equal Baſes CFT, AZB, are betwixt themſelves as their Sides (CF, AZ.) And, *Fig.* 23, 24. *l.* 12.

4. Thoſe conical Superficies (BAF, QZR) which are like, have betwixt themſelves a Proportion duplicate to that which is betwixt the Diameters of the Baſes. And, *Fig.* 20, 21 *l.* 12.

5. All conical Superficies whatſoever have betwixt themſelves a Proportion which is compounded of the Proportions of the Sides (BA, QZ) and of the Diameters (BF, QR) which are in the Baſes. And, The ſame Figure.

6. Thoſe which are equal have their Sides and the Diameters of the Baſes reciprocally proportional; and thoſe which have them ſo, are equal.

All which is demonſtrated from *Coroll.* 1. as above we deduced the *Corollaries* concerning the cylindrical Surface out of the firſt *Corollary* there.

7. Laſtly, we may meaſure a right conical Surface, if we multiply the Side FC by half the Circumference of the Baſe. As if the Side be of 5 Feet, the Circumference of the Baſe of 20; multiply 5 by 10, and there will ariſe 50 ſquare Feet for the conical Superficies. The Demonſtration is manifeſt from the ſame firſt Corollary. *Fig.* 25. *l.* 12.

O PROP.

PROP. XIV. Theorem.

Fig. 8, 9. of this. THE *Superficies of a right Cone is to the Base, as the Side (BC) is to (AC) the Radius of the Base.*

Between the Side BC and AC the Radius of the Base, let OL be a mean Proportional. Therefore the Proportion of BC to AC is duplicate to (*Defin.* 10. *l.* 5.) the Proportion of OL to AC. Now (by the 13th of this) a Circle of the Radius OL is equal to the conical Superficies CBD. But the Proportion of this to ACG the Base of the Cone is (by 2. *l.* 12.) duplicate to the Proportion of OL to AC; and consequently is the same with the Proportion of BC to AC. Therefore the Proportion of the Conical Superficies CBD is to the Base ACG, as BC is to AC. *Q. E. D.*

Corollaries.

Fig. 27. 1. THE *Superficies of a right Cone produc'd by an equilateral Triangle turn'd about the Perpendicular (KA) is double to the Base (QT).*

For the Side KB is equal to BD, and consequently double to the half AB, which is the Radius of the Base.

Fig. 24. 2. *The Superficies of a Cone produc'd by a right-angled equicrural Triangle (EBD) is to the Base, as in a Square the Diameter is to the Side.*

For the Perpendicular BA being drawn, the right Angle B (by 26. *l.* 1.) is bisected, and consequently ABD is half right. But ADB is also an half right Angle (by *Coroll.* 11. *pr.* 32. *l.* 1.) Therefore DA, BA, are (by 6. *l.* 1.) equal; and consequently BD is the Diameter of the Square AK, whereof AD is the Side. Now the same AD is the Semidiameter of the Base PT, seeing the Perpendicular AB (by 26. *l.* 1.) bisects ED.

From

From which, and this 14th the *Corollary* is manifest.

3. *The Superficies of the right Cylinder, (GK) is to the Superficies of the right Cone (GBN,) as the Side of the Cylinder is to half the Side of the Cone.* Fig. 24.

For the Superficies of the Cone GBN is to the Base MI, as the Side BN is to QN the Semidiameter of the Base (by the 14th of this;) that is, as half the Side BN is to the fourth Part of the Diameter GN. But the Base MI (by the 12th of this) is to the Superficies of the Cylinder GK, as the fourth Part of the Diameter is to NK, the Side of the Cylinder. By Equality of Proportion therefore the conical Superficies GBN is to the cylindrical Superficies GK, as half the Side of the Cone is to NK, the Side of the Cylinder. *Q. E. D.*

A Lemma *to what follows.*

IN a Triangle, as NPV let there be drawn QD parallel to NV. Fig. 10.

I say that the Rectangle under PN and NV is equal to the Rectangle under PQ, QD, together with the Rectangle under NQ and the two NV, QD put together.

Draw NA perpendicular to the Side NP, and equal to NV; and the Rectangle NO being compleated, let the Diameter PA be drawn. Then from Q let there be drawn QE parallel to NA, which may cut PA in B. Through B let CF be drawn parallel to NP. Because AN is equal to NV, it is manifest that QB also is equal to QD, (from *Coroll.* 1. *p.* 4. *l.* 6.) Therefore the Rectangle ON is the Rectangle PNV, and FQ is PQD. It remains that we prove that the Rectangles OB, EC, BN, are equal to the Rectangle under NQ, and the two NA, BQ; that is, to the Rectangle under NQ, and the two Lines NV, QD. But that is manifest; for the Rectangle under NQ and NA, QB, is equal (*per* 1. *l.* 2.) to these three Rectangles; that under NQ and CA (that is, the Space EC,) and that under NQ and NC (that is, the Space BN,) and that under

NQ and QB that is again the Space BN, and consequently the Space OB which (*per* 43. *l.* 1.) is equal to BN. The Proposition therefore is manifest.

PROP. XV. Theorem.

Fig. 11, 12. *IF a right Cone be cut by the Plain QSR parallel to NZO; I say, that the Circle GHM whose Radius GH is a mean betwixt Part of the Side NQ, and QD, NV (the Radius's of the Circles QSR, NZO) taken together; is equal to the conical Surface intercepted betwixt the parallel Circles QSR, NZO.*

Let GF be the mean betwixt PN and NV. Likewise let GK be the mean betwixt PQ and QD; and let there be described the Circles GFL, GKT. This (by the 13th of this) will be equal to the conic Superficies QPR, and the other to the Superficies NOP. The Rectangle PNV (by the *Lemma*) is equal to the Rectangle PQD, together with the Rectangle under NQ and NV QD, taken together. But because (by the Construction) GF is a mean proportional betwixt PN, NV; the Rectangle PNV is equal to the Square of GF (by 17. *l.* 6.) And because GK is (by the Construction) a mean betwixt PQ, QD, the Rectangle (by 17. *l.* 6.) PQD is equal to the Square of GK: And because GH by the Hypothesis is a mean betwixt QN, and QD, NV, taken together, the Rectangle (by 17. *l.* 6.) under QN, and QD, NV, taken together, is equal to the Square of GH. Therefore the Square of GF is also equal to the Square of GH, and to that of GK. Therefore seeing Circles are betwixt themselves (by 2. *l.* 12.) as the Squares of their Radius's, the Circle GLF will also be equal to the two Circles GKT, GHM taken together. But (by the 13th of this) the Circle GLF is equal to the conical Superficies NPO. Therefore the conical Superficies NPO is also equal to the two Circles GKT, and GHM. But QPR one Part of the Superficies NPO is (by the same) equal to the Circle GKT. Therefore the remaining Part, which is comprehended betwixt the parallel Circles ZZ, SS, is equal to the Circle GHM. *Q. E. D.*

A Lem-

A Lemma to what follows.

RIght Lines (BH, CG) which in the Circle intercept equal Arches (BC, HG) are parallel. *Fig.* 13.

For let CH be drawn. Because the Arches BC, HG are by the Hypothesis equal, the alternate Angles also (by 29. *l.* 3.) BHC, GCH, will be equal. Therefore (by 28. *l.* 1.) BH, and CG are parallel. *Q. E. D.*

PROP. XVI. Theorem.

LET there be inscribed in a Circle a regular Figure of an even Number of Sides, and let it be equilateral; let EB be drawn from the Extremity of the Diameter unto B, the end of the Side next to the Diameter: and let the right Lines BH, CG, DF, join the Angles which are equally distant from A. *Fig.* 13.

I say that the Rectangle contain'd under the Diameter AE, and the subtense EB, is equal to the Rectangle of one Side of the inscribed Figure AB, or BC, &c. and of all the joining Lines BH, CG, DF, taken together.

Draw CH, DG: Because BH, CG, DF intercept (by 26. *l.* 3.) equal Arches, BC, HG; CD, GF; these Lines (by the *Lemma*) will be parallel. By the same Argument BA, CH, DG, EF, are parallel. All the Triangles therefore (by 27, and 15. *l.* 1.) BAK, KHL, LCM, MGN, NDO, OFE, are equiangular. Therefore (by 4. *l.* 6.) as BK, is to KA, so is HK to KL; and as HK is to KL, so is CM to ML; and as CM to ML, so is GM to MN; and as GM is to MN, so is DO to ON; and as DO is to ON, so is FO to OE. Therefore (by 12. *l.* 5.) as one of the Antecedents, BK, is to one of the Consequents KA; so all the Antecedents BK, KH, CM, MG, DO, OF, (that is, all the joining Lines BH, CG, DF) are to all the Consequents AK, KL, LM, MN, NO, OE (that is, to the Diameter

AE.) But (by 8. *l.* 6.) as BK is to AK, so is EB to BA. Therefore as all these together BH, CG, DF are to AE, so is EB to BA. Therefore (by 16. *l.* 6.) the Rectangle under BA on one Part, and all the joining Lines BH, CG, DE, on the other, is equal to the Rectangle which is under AE and EB. *Q. E. D.*

PROP. XVII. Theorem.

Fig. 14. *LET there be inscribed in DAF a Segment of a Circle, whose Base DF is perpendicular to the Diameter AOE, a Figure equilateral, and of an even Number of Sides; and let there be drawn, as in the foregoing, the Line EB.*

I say, that the Rectangle comprehended under EB, and AO part of the Diameter, is equal to the Rectangle which is under one Side of the inscribed Figure, and all the joining Lines BH, CG, &c. taken together with DO half the Base.

The Demonstration is the same with that of the foregoing.

Lemma 1. *to what follows.*

Fig. 13. LET there be inscribed in the greatest Circle of a Sphere a regular Figure, which hath its Sides measured by the Number Four; and stands about the Axis AE; which Axis remaining unmov'd, let the Circle be turn'd round together with the Figure:

I say, that there will be inscrib'd in the Sphere a Body contained under conical Superficies.

It is manifest (see *Defin.* 2. *l.* 12.) that BA, HA, likewise DE, FE, describe entire Superficies of right Cones. Then because the Lines CB, GH, and GF, CD, being produced do concur on both Sides in the same Point of the Diameter AE which is in like manner to be drawn out, and cuts the joining Lines perpendicularly; It is also manifest that the said Lines CB,

CB, GH, *&c.* do describe Parts of right conical Surfaces which are intercepted betwixt the parallel Circles, which the Tops of the Angles B, C, D, design in the spherical Superficies.

Lemma 2.

LET DAF be the greatest Section of a Segment of a Sphere whose Axis is AO. Let there be inscribed in this a Figure having all the Sides equal, the Base excepted, and let it be turn'd round about the Axis AO.

I say, that a Body contain'd under conical Superficies will be inscrib'd in the spherical Segment.

This is proved as the foregoing *Lemma.*

PROP. XVIII. Theorem.

LET the same Things be suppos'd which were in the first Lemma *; and let the right Line (EB) be drawn from the Extremity of the Diameter unto the end of the Side next to the Diameter.* Fig. 13.

I say, that a Circle the Square of whose Radius (I) is equal to the Rectangle AEB contain'd under the Diameter AE, and the subtense EB, is equal to all the conical Surfaces inscrib'd in the Sphere.

That is, a Circle whose Radius (I) is a mean Proportional betwixt AE and EB.

Because the right Lines BH, CG, DF, are equal to the right Lines BK, CM, DO, taken twice; (by 1. *l.* 2.) the Rectangle under one Side of the Figure inscrib'd in the greater Circle (to wit, under AB, or BC, or CD, or DE,) and under all the joining Lines together BH, CG, DF, is equal to the Rectangle under AB and BK, together with that which is under BC and the Compound of BK and CM, together with that which is under CD and the Compound of CM and DO, together with that which is under DE and DO; for so each of the Lines BK, CM and DO, are taken

twice. But (by the 16th of this) the Rectangle under AB and all the joining Lines BH, CG, DF, taken together is equal to the Rectangle AEB; that is, (by the Hypothesis) to the Square of I. Therefore the Square of I is equal to the Rectangles under AB and BK, and under BC and the Compound of BK and CM, under CD and the Compound of CM and DO, and under DE and DO. Now let P be a mean proportional betwixt AB and BK; and Q a mean betwixt BC and the Compound of BK and CM; and R a mean betwixt CD and the Compound of CM, DO; S a mean betwixt DE and DO. The Squares therefore of P, Q, R, S, (by 17. *l.* 6.) are equal to the above-said Rectangles. Wherefore seeing I have already shew'd the Square of I to be equal to the same Rectangles, it must also be equal to the Squares of P, Q, R, S, together. Seeing therefore (by 2. *l.* 12.) Circles are betwixt themselves as the Squares of their Radius's; the Circle described of the Radius I will also be equal to all the Circles together whose Radius's are P, Q, R, S, (as is manifest from 22. *l.* 6.) But the Circles of the Radius's P and S are (by the 15th of this) equal to the conical Superficies which the Sides AB, ED, have produc'd, forasmuch as P is a mean Proportional betwixt AB the Side of the Cone, and BK the Radius of the Base; and S is a mean proportional betwixt ED and DO; and the Circle of the Radius Q is (by the 15th of this) equal to that Segment of a conical Superficies which is intercepted betwixt the two parallel Circles of the Diameters CG, BH, because Q is a mean betwixt BC and the Compound of BK, CM: And for the same Cause the Circle of the Radius R is equal to a Segment of a conical Surface, which is intercepted betwixt the parallel Circles of the Diameters CG, DF. Therefore the Circle described from the Radius I is equal to the conical Surfaces inscribed in the Sphere taken all together. *Q. E. D.*

PROP. XIX. Theorem.

Fig. 14. *LET the same Things be suppos'd which were in the 2d* Lemma, *and let the right Line EB*

EB be drawn from the Extremity of the Diameter (AE) to the end of AB the Side next to the Diameter.

I say, that a Circle whose Radius is a mean proportional betwixt (EB) and AO) the Axis of the Segment, is equal to all the conical Superficies inscribed in the spherical Segment DAF.

The Demonstration is altogether the same with that of the foregoing; only for *Prop.* 16. let *Prop.* 17. be cited.

PROP. XX. Theorem.

Conical Superficies inscrib'd in a Sphere do at length end in the Superficies of the Sphere. Fig. 15.

Let there be given a Superficies as small as you will, as X. It is manifest that within the spherical Superficies ACEG, there may be given some other Concentrical thereto, which falls short of this by a Quantity less than X. Let ACEG, DPLM, be the greatest Circles of both, as cut with a Plain through the Center. Let there be drawn the Diameter ADE, and in D let NQ touch it. If the Arch AE be bisected in C, and again the Remainder be bisected, and so on, there will be left at last the Arch AB (as is manifest of it self:) less than the Arch AN. If to this Arch the right Line AB be subtended, it is manifest that it will not reach to the Circumference PDML, and that it will be a Side of an equilateral Figure, of an even Number of Sides inscribed in the Circle CAGE, no Side whereof reacheth unto the Circumference PDML. Wherefore if all be turn'd round about the Diameter AE, it is manifest that there will be inscribed in the exterior Spherical Surface conical Surfaces, which include the spherical Surface, which is concentrical to the other, and consequently (by *Axiom* 3. of this) are greater. Because therefore the spherical Surface DPLM falls short of the spherical Surface ACEG. by a Quantity less than the given one X; much more will the conical Surfaces fall short of the said spherical Surface ACEG by a Quantity less than the given one X, and (by *Defin.* 6. *l.* 12.) consequently will end in the Superficies ACEG. *Q. E. D.*

PROP.

PROP. XXI. Theorem.

Fig. 17. *Conical Superficies inscribed in a spherical Segment DAF end in the spherical Superficies of the Segment it self.*

This may be demonstrated by the same Reasoning as the foregoing was.

PROP. XXII. Theorem.

Fig. 16. *It was demonstrated* Prop. 18. *that a Circle whose Radius is a mean Proportional betwixt the Diameter AE and the right Line EB, which is drawn from the Extremity of the Diameter unto the end of the Side AB next to the Diameter, is equal to all the conic Superficies inscrib'd in the Sphere.*

I say, that this Circle (see Def. 6. l. 12.*) ends at length in a Circle, whose Radius is AE the Diameter of the Sphere.*

For if more and more Sides be infinitely inscribed in the greatest Circle (which then being turn'd round about AE produce conical Superficies); it is manifest that the Side AB becomes at length less than any given right Line, and consequently that the Subtense EB approaches to the Diameter AE within a Distance less also than any given one; from whence it comes to pass that the Difference of those AE, DE, becomes likewise less than any given one. Therefore much more shall the mean proportional betwixt AE, BE, which is always greater than BE, differ from AE at length by a Defect less than any given one. Therefore the Circle also whose Semidiameter is a mean betwixt AE and BE, will at length differ from a Circle whose Semidiameter is AE, by a Defect less than any given one whatsoever; that is, will end *(Def. 6. l. 12.)* in it. *Q. E. D.*

This which is clear enough of it self, there is no need to demonstrate more operosely.

PROP.

PROP. XXIII. Theorem.

IT was demonstrated Prop. 19. *that a Circle whose Radius is a mean proportional betwixt EB and the Axis of the Segment AO, is equal to all the conical Superficies inscrib'd in the spherical Portion DAF:* Fig. 17.

I say, that this Circle ends in a Circle whose Radius is the right Line AD drawn from the Vertex of the Segment unto the Circumference of the Circle DQFN which is the Base of the Segment.

For because it now appears from the foregoing Demonstration that EB doth at length end in AE; it will also be manifest that the mean Proportional betwixt EB and AO doth at length end in the mean Proportional betwixt AE and AO, that is, (by *Coroll.* 2. *p.* 8. *l.* 6.) in AD it self. It is therefore manifest that the Circle also whose Radius is a mean Proportional betwixt EB and AO, doth end in the Circle of the Radius AD. *Q. E. D.*

A Lemma *to the following* Proposition.

IF the Diameter of one Circle be double to the Diameter of another, the one Circle will be fourfold to the other.

This is manifest from Prop. 2. *l.* 12. and *Defin.* 10. *l.* 5.

PROP. XXIV. Theorem.

THE Superficies of every Sphere is fourfold of the greatest Circle of the same Sphere. Fig. 16.

This most noble Theorem of *Archimedes* we shall from what goes before expeditiously demonstrate in this manner.

Let

Let an ordinate Figure, the Sides whereof are measured by the Number four, be understood to be inscrib'd in the greatest Circle of a Sphere about the Diameter A E. Let this Figure be turn'd round about A E, and so produce conical Surfaces inscrib'd in the spherical Surface, and let E B be drawn. It hath been demonstrated above (18. of this) that all conical Surfaces inscribed in a Sphere are equal to the Circle, the Square of the Radius whereof is equal to the Rectangle A E B, that is, whose Radius is a mean Proportional betwixt A E and E B. And this will always happen, although the Inscription be infinitely continued. Wherefore seeing the inscrib'd conical Surfaces (by 20. of this) will at length end in the spherical Surface, and the Circle whose Radius is a mean betwixt A E and E B will at length end (by 22. of this) in the Circle whose Radius is A E; the spherical Surface it self also (by 2. of this) will be equal to the Circle of the Radius A E, that is, (by the foregoing *Lemma*) to four Times the greatest Circle A C E G. *Q. E. D.*

He that shall read *Archimedes* will find that the Way here used in demonstrating this most noble Theorem is much shorter and clearer than that of *Archimedes.*

Corollary.

FRom this admirable Theorem, whereby *Archimedes* hath purchas'd to himself an immortal Name amongst the Geometricians, a Circle equal to a spherical Surface is obtain'd; that to wit, whose Semidiameter is the Diameter of a Sphere, or whose Diameter is double to the Sphere's Diameter.

Scholium.

WE are now well provided for the measuring of a spherical Surface, the chief amongst all Curve ones. And it is perform'd these two Ways.

1. Let the greatest Circle of the Sphere be measured (according to *Schol. Prop.* 6. of this) and let it be multiplied by 4. As, if the greatest Circle of the Orb of the Earth be found to contain 49,075,000 Square Miles, then accord-

according to this 196,325,000 ſquare Miles are contain'd in the whole ſpherical Surface.

2. The Diameter of a Sphere multiplied by the Circumference of the greateſt Circle gives you the ſpherical Superficies. According to which, if the Earth's Diameter conſiſt of 7,853 Miles, and conſequently the Circumference of the greateſt Circle conſiſts of 25,000, the whole ſpherical Surface will be in the ſame Miles 196,325,000; for 7,853 × 25,000=196,325,000.

The Demonſtration appears from *Coroll.* 1. *Prop.* 5. of this; for a Rectangle under the Diameter of a Sphere, and the Circumference of the greateſt Circle, is according to that *Corollary* fourfold of the greateſt Circle.

PROP. XXV. Theorem.

THE Surface of any ſpherical Portion whatever (as DAF) is equal to a Circle, whoſe Radius is the right Line (AD) drawn from the Vertex of the Portion to the Circumference of the Circle (DQFN) which is the Baſis of the Segment. Fig. 17.

Let a Figure equilateral and of an even Number of Sides, the Baſe being ſet aſide, be imagin'd to be inſcrib'd in the Section of the greateſt Portion about the Axis AO; this Figure being turn'd round about AO will inſcribe conical Surfaces in the Portion. Let the right Line EB be drawn alſo as above (in 18 and 19 of this.) All the conical Surfaces now inſcribed are equal (by the 19th of this) to the Circle whoſe Radius is a mean proportional betwixt EB and the Axis of the Segment AO. And this will always happen if the Inſcription be infinitely continued. Wherefore ſeeing both the conical Surfaces inſcrib'd in the Segment end at length (by 21 of this) in the ſpherical Surface of the Segment, and the Circle whoſe Radius is a mean betwixt EB and AO ends (by 23.) in the Circle of the Radius AD; the ſpherical Surface of the Portion alſo DAF (by 2.) will be equal to the Circle of the Radius AD. *Q. E. D.*

This is another of the more noble Inventions of *Archimedes*, which as the former we have demonſtrated in a much ſhorter and clearer Way than he did.

PROP.

PROP. XXVI. Theorem.

Fig. 28. *THE Superficies of a right Cylinder circumscrib'd about the Sphere (as the Cylinder HPSV) is equal to the Surface of the Sphere.*

And if a Cylinder and Sphere be cut by Plains perpendicular to the Axis (BG); each Segment of the Cylindrical Surface will be equal to each Segment of the Sphere.

Part I. Because the Side HP of the Cylinder is (by the Hypothesis) equal to PS the Diameter of the Base; the Cylindrical Surface HS will be (by *Coroll. p.* 12. of this) fourfold of the Base; that is, of the greatest Circle of the Sphere inscrib'd in the Cylinder; of which seeing (by 24 of this) the spherical Surface it self is also fourfold, this will be equal to the Cylindrical Surface. *Q. E. D.*

Part II. Let the right Lines BO, GO, be drawn. Because the Angle BOG (by 31. *l.* 3.) is right, as being the Angle in the Semicircle, and OC falls perpendicular from it upon BG; BO (by *Corol.* 2. *p.* 8. *l.* 6.) will be a mean proportional betwixt GB and BC, that is, betwixt IT and HI. Therefore the Circle of the Radius BO (by 11. of this) will be equal to the Cylindrical Surface HT. But the same Circle is also (by the foregoing) equal to the Segment of the spherical Surface OBK. Therefore the Cylindrical Surface HT and the spherical OBK are equal.

Then because it is shew'd in the same manner that the Cylindrical Surface HX is equal to the spherical QBR, the remaining Cylindrical Surface IX will be equal to the remaining spherical Surface QOKR, which is intercepted betwixt two parallel Circles.

And from these the Proposition is manifest of all Segments.

[Coroll. *Hence the Superficies of a Cylinder circumscrib'd about a Circle is double to the Bases.*]

PROP.

PROP. XXVII. Theorem.

THE Segments of a spherical Surface divided by parallel Circles have that Proportion amongst themselves, which the Segments (BC, CD, DA, AE, EF, FG) of that Diameter (BG) which is perpendicular to the parallel Circles have amongst themselves. Fig. 18.

It follows from the foregoing. For by that the Segments of the spherical Surface OBK, QOKR, MQRN, *&c.* are equal to the Cylindrical HT, IX, LN, *&c.* But these (by 13. *l.* 12.) have the same Proportion betwixt themselves which the Segments of the Axis BC, CD, DA, *&c.* have. Therefore those also have the same Proportion. *Q. E. D.*

Scholium.

FRom hence the Proportion of Zones and Climates betwixt themselves becomes known. For they are to one another as the Segments of the Axis, which are known from the Table of Sines.

From the same also we learn to measure the Segments of a spherical Surface. For because both the whole Surface of the Sphere is known from *Schol. Prop.* 24. and the Proportion of the Segments, the same as that of the Parts of the Axis, is also given; it is manifest that each of the Segments become known.

Now both the foregoing, and all the rest of the Theorems which follow, are altogether singular and admirable, and well worthy that those who are studious of Geometry should give all Diligence to understand them.

A Lemma *to the following.*

IF a Plain (QN) touch a Sphere in (O) a right Line (AO) from the Center to the Contact is perpendicular to the Plain. Fig. 19.

Let

Let QN the touching Plain and the Sphere be cut through the Contact with two Plains, which in the Sphere may produce the Circles OG, OD, but in the Plain QN the right Lines CO, IO, which shall touch the Circles in O. Therefore by 18. *l.* 3. AO is perpendicular to both IO and CO, and consequently by 4. *l.* 11. perpendicular to the Plain QN. *Q. E. D.*

PROP. XXVIII. Theorem.

Fig. 20, 22, 21.

Every Sphere is equal to a Cone (ZO) whose Altitude (KO) is equal to the Radius of the Sphere; and the Base (Z) equal to the Superficies of the Sphere.

Let some Polyedral Body be understood to be circumscribed about the Sphere, and let the solid Angles thereof be cut off by new Plains touching the Sphere. Which being done there will arise another Polyedral Body containing the Sphere, but less than the former, and consisting of more Angles, and having a Surface compounded of more tangent Plains in Number but less in Magnitude. If the solid Angles of this Polyedrum be again cut off by new tangent Plains, and the Angles of the third Polyedrum thence arising likewise, and so on for ever; it will come to pass at length that both the Polyedrum will exceed the Sphere by a solid less than any given one whatsoever; and the Surface thereof compounded of tangent Plains (which, as I said, are endlesly less in Magnitude, and more in Number than they were before) will exceed the spherical Surface also by a Plain less than any given one whatever. Both which Things, although they might be demonstrated, yet because they are of themselves manifest enough, I shall for Brevity's sake take for granted. These Things being thus stated, we proceed.

The Polyedrum now design'd is compounded of Pyramids, the common Top whereof is the Center of the Sphere, and the Bases are tangent Plains, which constitute the Surface of the Polyedrum. And because the right Lines drawn from the Center A unto the Contacts of each of the Plains, are (by the foregoing *Lemma*) perpendicular to each of the Plains; therefore the height of all the Pyramids, whereof the Polyedrum consists,

will be equal; to wit, AB the Radius of the Sphere. If therefore the Plain X be supposed equal to the Surface of the Polyedrum it self, and upon it there be erected a Pyramid at the Heighth MN, which is also equal to the Radius of the Sphere; It is manifest (by *6. l.* 12.) that all the abovesaid Pyramids, that is, the whole Polyedrum are equal to the Pyramid XN. After the same manner all the rest of the Polyedrums containing the Sphere, which from the perpetual Abscission of the solid Angles will arise one after another infinitely, are always equal to the Pyramids (represented by XN) the Altitudes whereof MN are the Radius of the Sphere; but the Bases (X) equal to the Surfaces of Polyedrums encompassing the Sphere. Wherefore, seeing at length both the Polyedrums (as I said above) do end in a Sphere, and the Pyramids, (XN) as I will shew by and by, do end in the Cone ZO; the Sphere also (by 1. of this) will be equal to the Cone. *Q. E. D.*

But that the Pyramids XN end in a Cone, I thus shew. The Surfaces of Polyedrums end in the Surface of the Sphere, as it was taken for granted above. But the Bases X of the Pyramids XN are always suppos'd equal to the Surfaces of the Polyedrums; and Z the Base of the Cone, ZO is by the Hypothesis equal to the Surface of the Sphere; therefore the Bases X also will end in the Base Z; and consequently seeing the Pyramids XN be to the Cone, which by the Hypothesis is of equal Height (by *Corol. Prop.* 11. *l.* 12.) as the Base X is to the Base Z, the Pyramids also will end in the Cone.

The Demonstration of this Proposition and the following is altogether diverse from that which *Archimedes* made use of, which indeed is very subtile and ingenious, but Prolix and difficult; to which there are premis'd two Positions that are manifest, and eleven Propositions, besides others not a few, on which they depend. But the Theorem it self, as propounded by *Archimedes*, is thus; Every Sphere is fourfold of a Cone, which hath a Base equal to the greatest Circle of the Sphere, and its Altitude equal to the Radius.

Scholium.

FRom this noble Theorem is deduc'd the Mensuration of the most noble of solid Figures. For if the sixth

Part of the Diameter, or the third Part of the Semidiameter be multiplied by the Surface of the Sphere, already known by *Schol. Prop.* 24. there will arise the Solidity of the Sphere.

Suppose the Superficies of the Earth be found to contain 196,325,000 square Miles, and let the third Part of the Semidiameter consist of 1309 such Miles. Multiply the two Numbers together, the Product 256,989,425,000 will be the Number of the cubic Miles of the Earth's Solidity.

For seeing a Sphere (by this *Prop.*) is equal to a Cone whose Altitude is the Radius of the Sphere, and its Base the Surface of the same Sphere, and the Solidity of the Cone (by *Schol. Prop.* 6. of this) is produc'd from the third Part of the Altitude (that is, of the Radius of the Sphere) multiplied by the Base (that is, the Surface of the Sphere,) the Sphere's Solidity also is obtain'd from the 3d Part of the Radius multiplied into the Superficies.

PROP. XXIX. Theorem.

Fig. 23. *EVery Sector of a Sphere is equal to a Cone whose Altitude is the Radius of the Sphere, and the Base the spherical Superficies of the Sector.*

First, let the Sector AECG be less than an Hemisphere. Let a right-lin'd polyedral Body be understood to be circumscrib'd about the Sector. Now if all the remaining Ratiocination be carried on after the same manner as was done in the foregoing, the Thing sought will be concluded in the same manner. This Thing alone will require to be shew'd, upon which indeed the whole Reasoning depends; to wit, that the Superfices of the Polyedrum, which is compounded of Plains on every Side, touching the Surface of the Sphere ECG, is greater than the Surface ECG. Which is done thus. Let another equal and like Surface be conceiv'd to be set to the Surface ECG encompass'd with touching Plains in the very same manner as the other is. There will now (by *Axiom* 3. of this) the whole Surface compounded of Plains, be greater than the whole spherical Surface. Therefore half the Surface compounded of Plains will also be greater than half the spherical Surface ECG.

Then

Then let the Sector AEBG be greater than an Hemisphere. Both Sectors taken together are (by the foregoing) equal to a Cone whose Heighth is the Radius of the Sphere, and its Basis the whole Superficies; that is, (by 11. *l.* 12.) to two Cones which have the same Heighth, but have their Bases equal to the Segments of the spherical Superficies ECG, EBG. But one of the Sectors AECG, that which is less than an Hemisphere, is by Part 1. equal to a Cone, whose Altitude is the Radius, and its Base the Surface ECG. Therefore the other Sector AEBG is equal to the other Cone whose Heighth is the Radius, and its Base the remaining spherical Surface EBG. *Q. E. D.*

Corollary.

SEeing (by 25. of this) the Superficies ECG is equal to the Circle of the Radius CG, and the Superficies EBG equal to the Circle of the Radius BG; the Sectors AECG, and AEBG, will be equal to Cones whose Altitude is the Radius of the Sphere, and their Bases Circles of the Radius's CG, and BG.

Scholium.

FRom these Things is deduc'd the measuring both of Sectors and Segments of Spheres; of Sectors (as appears from *Schol. Prop.* 6. of this) if the third Part of the Radius be multiplied by the spherical Surface of the Sectors, which is already known from *Schol. Prop.* 27. or by the Circle of the Radius CG or BG; and of Segments, if the Cone EAG be measured, and be taken away from the Sector, if it be less than an Hemisphere; but added thereto if it be greater. *Fig.* 23.

The Segment (MQRN) which lies betwixt two Circles, whether parallel or not parallel, is measured; if the Segments QBR and MBN already known, be substracted one out of the other. *Fig.* 18.

PROP. XXX. Theorem.

AN *Hemisphere* (*EOBD*) *is double to the Cone* (*EBD*) *which hath the same Base and Altitude with it self.* *Fig.* 24.

The Cone whose Basis is the hemispherical Superficies E O B D, and its Altitude the Radius A B, is to the Cone E B D (by 11. *l.* 12.) as Base is to Base; that is, as the hemispherical Surface E O B D is to the greatest Circle P T. Therefore, seeing the hemispherical Superficies E O B D is double to the greatest Circle (by 24. of this) the Cone also which hath the Superficies E O B D for its Base, and the Radius A B for its Altitude, is double to the Cone E B D. But (by 28. of this) the Hemisphere is equal to a Cone which hath the Radius for its Altitude, and the hemispherical Superficies for its Base. Therefore the Hemisphere is also double to the Cone E B D. *Q.E.D.*

PROP. XXXI. Theorem.

Fig. 25. *LET a Sphere be divided into two Segments ILBG, ISKG, by the Plain IQGT which doth not pass through the Center A; and let the Diameter BOK be perpendicular to the cutting Plain.*

As the Altitude O B of the Segment ILBG, is to the Radius of the Sphere AB: So let OK, the Altitude of the other Segment, be made to the other Line KN.

In like manner, As OK, the Altitude of the Segment ISKG, is to the Radius AK or AB, so let the Altitude OB of the other Segment be made to the other Line BD. Which Things being suppos'd, I say,

1. *The Cones ING and IDG whose Altitudes are ON, OD, and IQGT their common Base, are equal to the spherical Segments.*

2. *There is the same Proportion of the Segments as there is of the right Lines DO, NO.*

3. *The Segment ISKG is to the greatest Cone IKG inscribed in it, as NO is to KO; and the Segment ILBG is to the greatest Cone IBG inscrib'd in it, as DO is to BO.*

Part

Part I. Let the Sphere and Cones be cut by a Plain through the Diameter BK. There will be produced in the Sphere the greatest Circle BLKG, and in the Cones the Triangles BIG IKG. And because BOK the Diameter is (by the Hypothesis) perpendicular to the Circle QT, IOB (by *Def.* 3. *l.* 11.) will be a right Angle. The Angle in the Semicircle is also a right one (by 31. *l.* 3.) Because therefore in the Triangle BIK, there is drawn from the right Angle, IO perpendicular to the Base BK; BI will be to IO, as (by 8. *l.* 6.) BK to KI. Therefore the duplicate Proportion of BI to IO is equal to the duplicate Proportion of BK to KI; that is, (because BK, KI, KO (by *Corol.* 2. *Pr.* 8. *l.* 6.) are three Proportionals) equal to the Proportion of BK to KO.

Then because OB is (by the Hypothesis) to BD, as OK is to the Radius AB; by Inversion it will be always thus, DB is to BO, as AB to OK; and by Permutation thus, DB is to BA, as BO to OK; and by Compounding thus, DA is to BA, as BK is to OK. Because therefore I have already shew'd the Proportion of BK to OK to be duplicate to the Proportion of BI to IO, and consequently (by 2. *l.* 12.) equal to the Proportion betwixt the Circles describ'd by the Radius's BI, IO; DA will also be to BA, as the Circle of the Radius BI, to the Circle of the Radius IO. Therefore the Cone under the Altitude DA, and for the Base, the Circle of the Radius IO; that is, the Circle QT, is equal to the Cone under the Altitude BA, (by 15. *l.* 12.) which hath for its Base the Circle of the Radius BI; that is, (by *Corol.* 29. of this) the spherical Sector AIBG. Wherefore if the same Cone IAG be added as well to the Sector AIBG, as to the Cone under DA, and the Circle QT, the Wholes will be equal; to wit, the spherical Segment ILBG will be equal to two Cones, whereof one is that which is under the Base QT and the Altitude DA, and the other IAG is under the same Base QT, and the Altitude OA. But these two Cones (by 14. *l.* 12.) make up the Cone IDG. Therefore the Segment ILBG will be equal to the Cone IDG. *Q. E. D.*

By the same Reasoning the Segment ISKG will be equal to the Cone ING, with this only Change, that the Cone IAG which before was added, be now taken away.

Part II. This is manifest out of the first. For the Cones IDG and ING are betwixt themselves (by *p.* 14. *l.* 12.) as are DO and NO. Therefore the Segments also ILBG, ISKG, equal to those Cones, are betwixt themselves, as the right Lines, DO, NO.

Part III. This likewise is manifest from the first. For the Cone IDG is to the Cone IBG, (by the same) as DO is to BO. Therefore the Segment also ILBG which is equal to the Cone IDG, is to the Cone IBG, as DO is to BO.

Scholium.

FRom the first Part of this Proposition there arises another Way of Measuring spherical Segments, and that a very easy one; if, to wit, the Cones IDG, ING, be measured; which will be done if the third Parts of the right Lines DO, NO, be drawn into the Circle QT.

PROP. XXXII. Theorem.

Fig. 24. A *Right Cylinder (GK) is both in Solidity and the whole Superficies to the Sphere about which it is circumscrib'd, as* 3 *to* 2.

Let BQ be the common Axis of the Sphere and Cylinder, and EBD the greatest Cone inscrib'd in the Hemisphere EOBD. Because the Cylinder EK (half of GK) is (by 10. *l.* 12.) treble to the Cone EBD, while the Hemisphere is double to the same Cone, (by 30. of this) it is manifest that the Cylinder EK is to the Hemisphere as 3 to 2. Therefore also the whole Cylinder GK is to the whole Sphere QEBD as 3 to 2. Which was the first Part.

Then because the Side of the Cylinder KN is equal to GN the Diameter of the Base, its Superficies without the Bases will be fourfold (by *Corol. Pr.* 12. of this) of the Base MI, and consequently taken together with the Bases, that is, the whole Superficies of the Cylinder, will be sixfold of the Base MI, which is equal to the greatest Circle of the Sphere. But the Superficies of the Sphere is fourfold of that greatest Circle. Therefore the whole Superficies of the Cylinder GK is to the Superficies of the Sphere, as 6 to 4, or as 3 to 2. Which was the other Part.

There-

Therefore a Cylinder is both in Solidity and the whole Superficies to the Sphere, about which it is circumscrib'd, as 3 to 2. *Q. E. D.*

Scholium.

IT is an Argument what a great Value *Archimedes* puts upon this Theorem, that he would have a Sphere inscrib'd in a Cylinder set upon his Tomb. And perhaps amongst so many other Famous Discoveries, this chiefly and above all others pleas'd him for this reason, to wit, because there was one and the same rational Proportion both of Bodies, and of the Surfaces which contain them. We have demonstrated a like Identity of Affections betwixt Rings, and the Surfaces of Rings, in the 4th Book of our Cylindricks and Annularies, *Prop.* 13, 14, 15. And another Famous Example of the same hath also offer'd it self to me in the Sphere it self. For I have found, that like as a Sphere is to a right Cylinder which encompasseth it (which will necessarily be equilateral) as 2 is to 3, and this both in respect of Solidity and Surface; so likewise the Sphere hath to an equilateral Cone encompassing it, that Proportion which 4 hath to 9; and this both in regard of Solidity and Superficies. From which this also follows, That the sesquialteral Proportion found by *Archimedes* in the Sphere and Cylinder, is continued in three Solids, Sphere, Cylinder, and equilateral Cone. The Demonstration of both which Things, with some other Theorems of my own, in which the wonderful Nature of the Sphere will more appear, I shall subjoin in the thirteen following Propositions.

PROP. XXXIII. Theorem.

THE Superficies of a Sphere is double to the Superficies of a square Cylinder inscrib'd in the same Sphere. Fig. 26.

Let A K L be the Square inscrib'd in the greatest Circle of a Sphere, from which turnd round, there is described a square Cylinder; and let A L be drawn as a Diameter common to the Cylinder and Sphere. Because the Square A L is (by 47. *l.* 1.) equal to the equal Squares A K,

KL, it will be double to one AK. Therefore also the Circle of the Diameter AL, is (by 2. *l.* 12.) double to the Circle, whose Diameter is AK; to wit, to the Circle CN. But the Superficies of the Sphere is (by 24. of this) fourfold to the Circle whose Diameter is AL; for that is the greatest Circle of the Sphere, seeing AL is the Diameter of the Sphere. Therefore the Superficies of the Sphere is Eightfold of the Circle CN. But because LK, KA (by the Hypothesis) are equal, the cylindrical Superficies ACL is (by *Corol. Pr.* 12. of this) quadruple of the Circle CN. Therefore since the Superficies of the Sphere is Eightfold of the same Circle, it will be double to the cylindrical Superficies. *Q. E. D.*

PROP. XXXIV. Theorem.

Fig. 26. *THE Superficies of a Sphere hath that Proportion to the whole Superficies of a square Cylinder inscrib'd in it, which* 4 *hath to* 3.

Let the same Things be suppos'd which were in the foregoing Demonstration. Because by the Hypothesis LK the Side of the Cylinder, and AK the Diameter of the Base thereof are equal, the cylindrical Superficies CL will be quadruple (by *Corol. Pr.* 12. of this) to the Base CN, and consequently the whole Superficies of the Cylinder is to both Bases CN and SL, as 6 is to 2. But the Superficies of the Sphere is to both Bases together CL, SL, as 8 is to 2, seeing in the foregoing it was shew'd that it is to one Base as 8 to 1. Therefore the Superficies of the Sphere is to the cylindrical Superficies CL as 8 is to 6, or 4 to 3. *Q. E. D.*

Corollary.

THE whole Superficies of a right Cylinder describ'd about a Sphere, is to the whole Superficies of an equilateral Cylinder inscrib'd, as 2 is to 1. For the circumscrib'd is to the spheric Superficies as 12 is to 8 (by 32. of this.) But the Spheric is to the inscrib'd as 8 is to 6 by this present Proposition. Therefore the circumscrib'd is to the inscrib'd as 12 is to 6, or 2 to 1.

PROP.

PROP. XXXV. Theorem.

A *Portion of any ſpherical Superficies whatever (as ILBG) hath the ſame Proportion to the Superficies of the greateſt inſcribed Cone, which (BG) the Side of the Cone hath to (GO) the Radius of the Baſe.* Fig. 26, or 25.

Becauſe (by 25. of this) the Superficies of the Portion ILBG is equal to the Circle of the Radius BG; the Proportion thereof to QT, that is, to the Baſe of it ſelf and of the Cone, will be duplicate to the Proportion (by 2. *l.* 12.) of BG to GO; that is, (by 14. of this) of the Proportion of the conical Superficies IBG to the ſame Baſe QT. Therefore it is manifeſt (by *Def.* 10. *l.* 5.) that the Superficies ILBG is to the conical Superficies IBG, as the ſame conical Superficies IBG, is to the Baſe QT. Wherefore ſeeing the conical Superficies IBG, is to the Baſe QT, as BG (by 14. of this) is to GO, the Superficies of the Portion will alſo be to the conical Superficies IBG inſcrib'd in it, as BG is to GO. *Q.E.D.*

PROP. XXXVI. Theorem.

THE *Superficies of the Hemiſphere (EOBD) hath that Proportion to (EBD) the Superficies of the greateſt right inſcribed Cone, which in a Square the Diameter hath to a Side; and that Proportion to the Superficies of a like Cone circumſcribed, as the Side in a Square hath to the Diameter.* Fig. 24.

I. The Demonſtration of the firſt Part is manifeſt from the foregoing. For EOBD the Superficies of any Portion whatever, and conſequently of the Hemiſphere, is to the conical Superficies inſcrib'd, as BD is to DA. But BADK is a Square whoſe Diameter is BD and the Side DA.

Part II. Let EBC be half of the Square circumſcrib'd about the Circle (whoſe Center is A); which EBC being turn'd about the Axis AB, let there from thence be produc'd a Cone circumſcribed about the Hemiſphere. Now becauſe the Square EC is (by 47. *l.* 1.) double to the Square EB or GI, the Circle of the Diameter EC Fig. 6. *l.* 4.

also

alſo is (by 2. *l.* 12.) double to the Circle whoſe Diameter is GI, that is, to the Circle HGDI. But (by 24. of this) the Superficies of the Hemiſphere included in the Cone EBC is double to the ſame Circle. Therefore the Circle of the Diameter EC is equal to the hemiſpherical Surface. Wherefore ſeeing the conical Superficies EBC is (by 14. of this) to the Circle of the Diameter EC, to wit, to its own Baſe, as the Side BE is to EA the Radius of the Baſe; it will be alſo to the hemiſpherical Superficies inſcribed in it, as BE is to EA; that is, as the Diameter in a Square is to a Side. *Q. E. D.*

PROP. XXXVII. Theorem.

The ſame Figure, with *Fig.* 13. *l.* 5.

A *Sphere hath the ſame Proportion to a ſquare conical Rhombus circumſcribed about it both in reſpect of Solidity and Surface; Which in a ſquare the Side hath to the Diameter.*

Let the ſquare EBCF be circumſcrib'd about HGDI, the greateſt Circle of a Sphere, from which Square as turn'd round about the Axis BF, let a conical Rhombus encompaſſing the Sphere be produc'd.

As EB a Side of the Square (ſee *Fig.* 6. *l.* 4.) is to the Diameter EC, even ſo let S be made to R; (ſee *Fig.* 13. *l.* 5.) and let this Proportion be continued through four Terms, S, R, Q, O, the Proportion then of S to O will be triplicate to the Proportion of S to R; that is, of EB to EC, and the Proportion of O to R will be duplicate to the Proportion of O to Q, or of R to S; that is, of EC to EB; and conſequently (by 20. *l.* 6.) O is to R as the Square of EC is to EB; from whence (by *Schol. Pr.* 6. and 7. *l.* 4.) O is double to R. Theſe Things being thus ſettled, let the Sphere EBCF be underſtood to be circumſcribed about the conical Rhombus. Thus the Sphere HGDI will be to the Sphere EBCF (by 18. *l.* 12.) in the triplicate Proportion of the Diameter GI or EB to the Diameter EC; that is, (as I have already ſhew'd) it will be as S to O. But the Sphere EBCF is to the conical Rhombus inſcribed in it (by 30. of this) as 2 is to 1; that is, (as I have ſhew'd above) as O is to R. Therefore by Equality of Proportion, the Sphere HGDI is to the ſame Rhombus which is deſcribed about it, as S is to R; that is, as in a Square the Side EB is

to

to the Diameter E C. Which was the first Part. Then from the second Part of the foregoing, it appears that the Superficies of the Hemisphere is to the Superficies of the Cone E B C, and consequently the Superficies of the whole Sphere is to the Superficies of the whole Rhombus E B C F, as in a Square the Side is to the Diameter. Therefore the Sphere as well in Solidity as in Superficies is to the square Rhombus E B C F, as in a Square the Side is to the Diameter. *Q. E. D.*

PROP. XXXVIII. Theorem.

THE Superficies of the Portion (BGKD) which contains an equilateral Cone (BKD) is double to the Superficies of the same Cone. Fig. 27.

This is manifest from 35. For the Superficies of the Portion B G K D is to the inscrib'd conic Superficies (by 35 of this) as B K is to B A. But because the Cone B K D is suppos'd to be equilateral, K B is equal to B D, and consequently double to B A. Therefore the Superficies B G K D is also double to the inscribed conical Superficies. *Q. E. D.*

PROP. XXXIX. Theorem.

THE Superficies of a Sphere is to the whole Superficies of an equilateral Cone inscribed in it, as 16 is to 9. Fig. 27.

Let Z be the Center of the Sphere, and B K D the equilateral Cone inscribed, and K Z A Q the Axis common to the Sphere and Cone. If the Sphere and Cone be cut through this, there will be produced in the Sphere the greatest Circle, and in the Cone, the equilateral Triangle B K D, one Side whereof will be the Diameter of the Basis of the Cone Q T. And because the Axis of the Cone K A is perpendicular to the Base Q T, B A K (*Def.* 3. *l.* 11.) will be a right Angle. Therefore the Square of B A is equal to the Rectangle K A O. (*Corol.* 1. *Pr.* 17. *l.* 6.) Now because the Side of the equilateral Triangle cuts off (*Corol.* 5. *Pr.* 15. *l.* 4.) a 4th Part of the Axis A O, the Rectangle K A O, that is, the Square of B A, will be treble to the Square of A O (by 1. *l.* 6.) Wherefore seeing the Square of the Radius Z O is (*Corol.* 3. *Pr.* 1. *l.* 2.) quadruple of the Square of A O, the Square of the Radius Z O will

will be to the Square of the Radius B A as 4 is to 3. Therefore the Circle O B K D is also (by 2. *l.* 12.) to the Circle Q T as 4 is to 3. Therefore four Circles O B K D, that is, (by 24. of this) the whole spherical Superficies D G is to the Circle Q T as 16 is to 3. But (*Corol.* 1. *Pr.* 14. of this) the Superficies of the equilateral Cone B K D is to the Circle Q T, to wit, its own Base, as 2 is to 1, and consequently the whole Superficies of the Cone B K D; that is, including its Base, is to the Base, to wit, the Circle Q T, as 3 is to 1, or 9 to 3. Therefore seeing I have shew'd that the Superficies of a Sphere is to the same Circle, as 16 is to 3, the Superficies also of the Sphere D G will be to the whole Superficies of the equilateral Cone as 16 is to 9. *Q. E. D.*

Or otherwise thus.

BEcause by *Corol.* 5. *Pr.* 15. *l.* 4. the Side B D of the equilateral Triangle cuts off a 4th Part of the Axis A O, the spherical Superficies B O D will be a 4th Part by 27 of this, and consequently the Superficies B G D K three 4th Parts of the Superficies of the whole Sphere. Wherefore if the whole Superficies be suppos'd to be 16, the Superficies B G K D will be 12. But (by the foregoing) the Superficies B G K D is double to the conical Superficies B K D, and consequently is to it as 12 to 6. Therefore the whole Superficies of the Sphere is to the conical B K D as 16 is to 6. Then because the Superficies of the Cone B K D (as being equilateral) is (by *Corol.* 1. *Pr.* 14. of this) double to the Base Q T, it is manifest that the conical Superficies B K D (to wit, without the Base) is to the whole Superficies of the Cone as 2 is to 3; that is, as 6 to 9. Therefore by Equality of Proportion the whole Superficies of the Sphere is to the whole Superficies of the equilateral Cone inscrib'd as 16 to 9. *Q. E. D.*

PROP. XL. Theorem.

Fig. 28. THE *Superficies of a Sphere bears the Proportion to the whole Superficies of an equilateral Cone circumscrib'd about it, that* 4 *doth to* 9.

Let there be circumscrib'd about the greatest Circle of a Sphere B P M the equilateral Triangle D O F, by which as turn'd round about the Axis O A B, let there be produc'd an equilateral Cone, circumscrib'd about the Sphere. And let there also be circumscrib'd about the equilateral

Tri-

Triangle DOF the Circle NDLOF which, as is manifest, is concentrical to the former; and let the Axis OAB be produc'd to N. Because BN is a 4th Part of the Axis ON as is manifest (from *Corol.* 5. *Pr.* 15. *l.* 4.) ON is double to KB. Wherefore the Proportion betwixt Circles being duplicate (by 2. *l.* 12.) of the Proportion of the Diameters, the Circle BPM will be to the Circle NDLOF as 1 to 4. But it hath already been shew'd in the first foregoing Demonstration, that the Circle NDLOF is to the Circle QT the Base of the equilateral Cone inscrib'd in the Sphere FL, as 4 is to 3. Therefore by Equality of Proportion the Circle BPM is to the Circle QT as 1 is to 3. But the whole Surface of the Cone DOF is (by *Cor.* 1. *Pr.* 14. of this) treble to QT. Therefore the whole Superficies of the Cone is ninefold of the Circle BPM. Wherefore seeing the Superficies of the Sphere TP is quadruple (by 24 of this) of the same Circle BPM, the whole Superficies of the equilateral Cone DOF is to the Superficies of the Sphere to which it is circumscrib'd, as 9 is to 4. *Q. E. D.*

[Coroll. 1. *From this Demonstration it is manifest that the Axis BO of an equilateral Cone circumscrib'd about a Sphere, is one and an half of the Diameter of the Sphere EK, or as* 3 *to* 2.

2. *That QT the Base of the Cone DOF is also one and an half of both Bases of the Cylinder circumscrib'd about the same Sphere. For QT is to BPM as* 3 *to* 1. *Therefore QT is to BPM twice as* 3 *is to* 2.

3. *That the Superficies of the Cone DOF is one and an half of the Superficies of the equilateral Cylinder circumscrib'd about the same Sphere. For that* † *is double to QT, while this is quadruple to BPM* *. *Therefore the conical Superficies will be to the Cylindrical as twice* 3 *to four times* 1; *that is, as* 6 *to* 4, *or as* 3 *to* 2.]

[†] *Per Corol.* 1. *p.* 14. of this.
* 24, and 26. of this.

PROP. XLI. Theorem.

Fig. 28.

THE whole Superficies of an equilateral Cone circumscrib'd about a Sphere, is quadruple to the whole Superficies of a Cone inscribed in the same Sphere.

By the foregoing the whole Superficies of the equilateral Cone DOF circumscrib'd, is to the Superficies of the Sphere as 9 to 4; and the Superficies of the Sphere is to the whole Superficies of the inscribed Cone SKT as 16 to 9 (by 39 of this.) Therefore by Perturbation of

Equality of Proportion the whole Superficies of the circumscribed equilateral Cone is to the whole Superficies of the equilateral inscribed, as 16 is to 4, or as 4 to 1. *Q.E.D.*

PROP. XLII. Theorem.

Fig. 29. *A Sphere hath that Proportion to BKC an equilateral Cone inscribed in it, which 32 hath to 9.*

Let the Sphere and Cone be cut by a Plain passing through the Common Axis KO, producing in the Sphere the greatest Circle OFKI, and in the Cone the equilateral Triangle BKC. Then a Plain being drawn thro' the Center A perpendicular to OK, let the Hemisphere FGKI be cut off, in which let the greatest Cone FKI be understood to be inscribed. Now because (by *Cor.* 5. *p.* 15. *l.* 4.) the Side BC of the equilateral Triangle cuts off a 4th Part of the Axis OK, PK will be to AK as 3 to 2, that is, as 9 to 6. But the Base QT is to the Circle OFKI, that is, to the Base ND, as 3 to 4, that is, as 6 to 8, as appears from what was demonstrated *pr.* 39. Wherefore seeing the Proportion of the Cone BKC to the Cone FKI is (by *Schol.* 2. *pr.* 15. *l.* 12.) compounded of the Proportion of the Altitude PK to the Altitude AK (that is, of the Proportion of 9 to 6) and of the Proportion of the Base QT to the Base ND (that is, of the Proportion of 6 to 8) the Cone BKC will be to the Cone FKI as 9 to 8. Wherefore seeing (by 30 of this) the Sphere CG is quadruple of the Cone FKI, the equilateral Cone BKC will be to the Sphere CG as 9 to 32. *Q. E. D.*

PROP. XLIII. Theorem.

Fig. 28. *AN equilateral Cone circumscrib'd about a Sphere, is eightfold of an equilateral Cone inscrib'd in the same Sphere.*

Let SKT and DOF be the equilateral Cones inscrib'd and circumscrib'd, and let OKB be the common Axis. Then let as well both the Cones as the Sphere be cut by a Plain passing through the Axis; their Sections will be two equilateral Triangles, and the greatest Circle BPM. About the Triangle DOF likewise let there be understood to be describ'd the Circle NDOF, by let the Axis OKB be produc'd unto N. Now because the Side DF of the equilateral Triangle doth (by *Corol.* 5. *pr.* 15. *l.* 4.) cut off NB the 4th Part of the Axis ON, it is manifest that ON is double to BK. In like manner, because the Side ST

of

of the other equilateral Triangle cuts off BC the 4th Part of the Axis BK, NO will be to BO, as BK is to CK; and by changing, as NO is to BK so is BO to CK. But NO is double to BK. Therefore BO is likewise double to CK. Therefore because of the Similitude of the Triangles, DOF, SKT, DF and ST also, to wit, the Diameters of the conical Bases will (by 4. *l.* 6.) be in a double Proportion betwixt themselves. Wherefore seeing the Cones DOF, SKT, be like, and consequently (by 12. *l.* 12.) their Proportion is triplicate to the Proportion of the Diameters DF and ST; which is that of 2 to 1, the Cone DOF will be to the Cone SKT as 8 to 1. *Q. E. D.*

PROP. XLIV. Theorem.

A *Sphere hath the same Proportion both in respect of Solidity and Surface to the equilateral Cone DOF circumscrib'd about it, which* 4 *hath to* 9. *Fig.* 28.

The Sphere TP is (by 42. of this) to the equilateral Cone SKT inscribed in it, as 32 is 9. But (by the foregoing) SKT the equilateral Cone inscribed is to DOF the equilateral Cone circumscribed as 1 is to 8, that is, 9 to 72. Therefore by Equality of Proportion the Sphere TP is to DOF the equilateral Cone circumscrib'd as 32 is to 72, that is, as 4 to 9. But in *Prop.* 40. we demonstrated that the Superficies of a Sphere is to the whole Superficies of an equilateral Cone circumscribed as 4 is to 9. Therefore a Sphere both in Solidity and Superficies is to an equilateral Cone circumscrib'd about it, as 4 is to 9. *Q. E. D.*

That therefore which *Archimedes* was surpris'd at in a Sphere and Cylinder encompassing it, we have also now demonstrated in a Sphere and an equilateral Cone encompassing, to wit, that there is the same rational Proportion of the Solidities betwixt themselves, which there is of the Surfaces. For as he found that the Sphere is to the Cylinder as well in Solidity as Superficies, as 2 to 3; so we have now taught, that the Sphere is in respect both of Solidity and Surface to an equilateral Cone encompassing, as 4 to 9.

But from hence we shall without much labour demonstrate that the very Proportion; to wit, the sesquialteral, which *Archimedes* shew'd to be betwixt the Sphere and Cylinder, is continued by the equilateral Cone circumscrib'd both in the Solidity and Superficies; and so we shall put an End to the present Work.

PROP.

PROP. XLV. Theorem.

See the Figure prefixed to this Treatise.

AN equilateral Cone circumscrib'd about a Sphere, and a right Cylinder in like manner circumscrib'd about the same Sphere, and the same Sphere it self, continue the same Proportion; to wit, the sesquialteral as well in respect of the Solidity, as of the whole Superficies.

For by 32. of this Book, the right Cylinder GK encompassing the Sphere, is to the Sphere, as well in respect of Solidity as of the whole Superficies, as 3 is to 2, or as 6 to 4. But by the foregoing the equilateral Cone BAD circumscrib'd about the Sphere is to the Sphere in both the said Respects, as 9 is to 4. Therefore the same Cone is to the Cylinder, both in respect of Solidity and Surface, as 9 is to 6. Wherefore these three Bodies, a Cone, Cylinder, and Sphere are betwixt themselves as the Numbers 9, 6, 4, and consequently continue the sesquialteral Proportion. *Q. E. D.*

[*PROP. XLVI.*

THE same sesquialteral Proportion holds betwixt an equilateral Cone and Cylinder circumscrib'd about the same Sphere, in respect of their whole Surfaces, their simple Surfaces, their Solidities, Altitudes and Bases.

This Proposition is manifest as to the whole Surfaces and Solidities from the foregoing; as to the simple Surfaces, from Coroll. 3. Pr. 40. *of this; as to their Altitudes and Bases, from* Coroll. 1, *and* 2, *of the same* 40*th Proposition.*]

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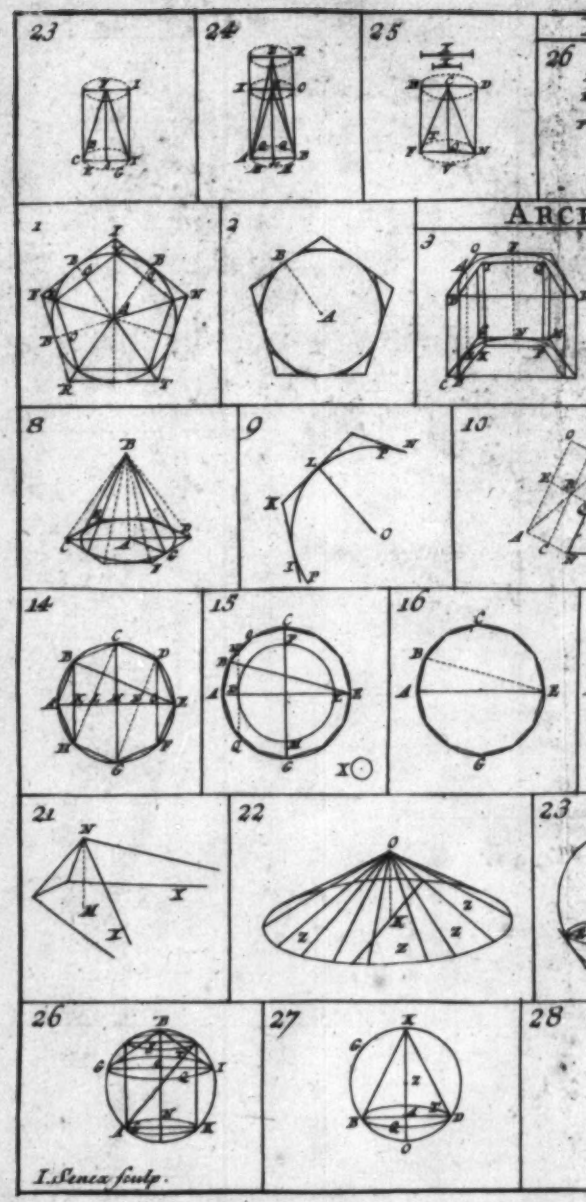
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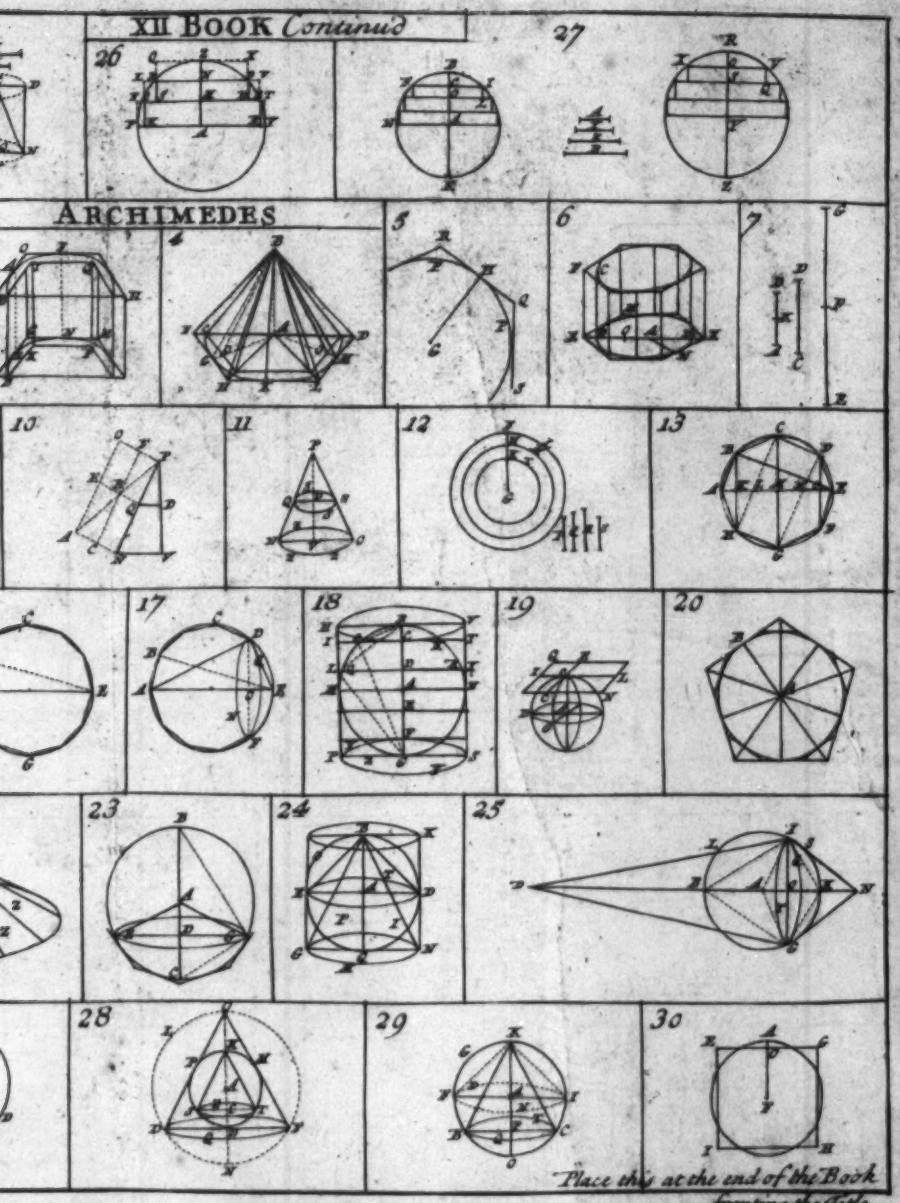

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